The Characterisation of Unital C*-algebras via Archbold Constants

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Abstract

Let A be a unital C*-algebra. For a in A, let $\mathrm{ad}_A\, a$ denote the inner derivation induced on A by a, and let $d\,(a, Z\,(A))$ denote the distance from a to the centre of A. Define $K\,(A)$ and $K_s\,(A)$ to be the least elements of $[0, \infty]$ such that, for all a in A,

$$d\,(a, Z\,(A)) \leq K\,(A)\, \|\mathrm{ad}_A\, a\|$$

and, for all a in A_{sa},

$$d\,(a, Z\,(A)) \leq K_s\,(A)\, \|\mathrm{ad}_A\, a\|\,.$$

An exposition of Dr Somerset's investigation of these constants is given. In particular, a theorem connecting the values of $K\,(A)$ and $K_s\,(A)$ to certain intersection properties of the primitive ideals of A is proved. The connection between this work and related results is briefly discussed.

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Contents

Chapter 1

Introduction

Let A be an algebra. A linear map D from A to A is said to be a **derivation** if, for all a and b in A, it satisfies the Leibnitz equation

$$D(ab) = D(a)\,b + aD(b)\,.$$

For a in A, define the adjoint map $\operatorname{ad}_A a$ from A to A by

$$(\operatorname{ad}_A a)(b) = [a, b] = ab - ba$$

for all b in A. It is easy to check that $\operatorname{ad}_A a$ is a derivation. Derivations of this form are known as **inner derivations**.

Let A be a Banach algebra. Then, for all a and b in A,

$$\|(\operatorname{ad}_A a)(b)\| = \|ab - ba\| \leq 2\|a\|\|b\|\,.$$

Hence $\operatorname{ad}_A a$ is bounded and

$$\|\operatorname{ad}_A a\| \leq 2\|a\|$$

for all a in A. Let $d(a, Z(A))$ be the distance from a to the centre $Z(A)$ of A. For z in $Z(A)$, $\operatorname{ad}_A a$ equals $\operatorname{ad}_A (a+z)$. Thus, for all a in A,

$$\|\operatorname{ad}_A a\| \leq 2d(a, Z(A)) \tag{1.1}$$

In [33], Stampfli showed that, when A is a primitive unital C*-algebra, equality holds in 1.1 for all a in A. In particular, equality holds when A is $B(H)$, the C*-algebra of bounded operators on a Hilbert space H. The results of [33] are essential for the further development of the theory, so an exposition is given in Chapter 3. Zsidó used the work of Stampfli in [36] to show that equality holds in 1.1 when A is a von-Neumann algebra. In [31], Somerset generalised Zsidó's result. The details are given in Chapter 6.

However, there are C*-algebras where the inequality 1.1 is strict. To study the inequality further, Archbold [3] introduced the constants $K(A)$ and $K_s(A)$, defined to be the least elements of $[0, \infty]$ such that, for all a in A,

$$d(a, Z(A)) \leq K(A)\|\operatorname{ad}_A a\|\,,$$

and, for all a in A_{sa},

$$d(a, Z(A)) \leq K_s(A)\|\operatorname{ad}_A a\|\,,$$

where A is a C*-algebra and A_{sa} is its self adjoint part. Additional constants can be analogously defined for other subsets of A. We see that, for all a in A,

$$\begin{aligned} d(a, Z(A)) &\leq d\left(\tfrac{1}{2}(a^* + a), Z(A)\right) + d\left(\tfrac{i}{2}(a^* - a), Z(A)\right) \\ &\leq K_s(A)\left\|\operatorname{ad}_A \tfrac{1}{2}(a^* + a)\right\| + K_s(A)\left\|\operatorname{ad}_A \tfrac{i}{2}(a^* - a)\right\| \\ &\leq \tfrac{1}{2}K_s(A)\left(2\|\operatorname{ad}_A a^*\| + 2\|\operatorname{ad}_A a\|\right) \\ &= 2K_s(A)\|\operatorname{ad}_A a\|\,. \end{aligned}$$

Therefore,

$$0 \leq K_s(A) \leq K(A) \leq 2K_s(A).$$

Since A_{sa} linearly generates A, if $K_s(A)$ is zero, A is commutative. If A is non-commutative then there exists a in A such that

$$0 < \|\mathrm{ad}_A\, a\| \leq \left\|\mathrm{ad}_A\, \tfrac{1}{2}(a^* + a)\right\| + \left\|\mathrm{ad}_A\, \tfrac{i}{2}(a^* - a)\right\|,$$

and hence $\|\mathrm{ad}_A\, b\|$ is non-zero for some b in A_{sa}. Furthermore,

$$\|\mathrm{ad}_A\, b\| \leq 2d(b, Z(A)) \leq 2K_s(A)\|\mathrm{ad}_A\, b\|,$$

and

$$K(A) \geq K_s(A) \geq \tfrac{1}{2}.$$

Clearly, when A is a non-commutative C*-algebra, $K(A)$ and $K_s(A)$ have the value $\frac{1}{2}$ if and only if equality holds in 1.1 for all a in A or for all a in A_{sa} respectively. It was shown by Kadison, Lance and Ringrose in [18], Theorem 5.3, that the set of inner derivations of A is norm closed in the set of derivations of A if and only if $K(A)$ is finite. Archbold [3] studied the stability of $K(A)$ and Archbold [3] and Batty [8] investigated $K(A \otimes_\beta B)$ where $A \otimes_\beta B$ is a C*-tensor product of C*-algebras A and B. In [32], Somerset showed that $K_s(A)$ is always of the form $\frac{n}{2}$, for some natural number n, or infinity. These results are described in more detail in Chapter 6.

In this paper we give an exposition of Somerset's work ([30], [31], [32]) on characterising C*-algebras by the values of $K(A)$ and $K_s(A)$. For simplicity, attention is restricted to the unital case. The reader is assumed to be familiar with basic C*-algebra definitions and facts, such as might be given in a first course.

In the second chapter of this paper we review some standard facts about representations of a unital C*-algebra A. We introduce $\mathrm{Spec}\, A$, the spectrum of A, and $\mathrm{Id}\, A$, the set of closed two-sided ideals of A. The subsets $\mathrm{Prim}\, A$, $\mathrm{Glimm}\, A$ and $\mathrm{Primal}\, A$ of $\mathrm{Id}\, A$ are defined and some of their topologies discussed. A new, easy proof of Proposition 4.5 of [4] is given (Proposition 2.9).

In the third chapter we give an exposition of the main results of [33]. It is shown that each element a of A has a closest scalar $\lambda(a)$ which satisfies the Pythagorean relation for operators (Theorem 3.4). The proof given is a modification of the proof in [33] and does not seem to have been given before. The circumcircle of the spectrum is introduced and used, with the functional calculus, to find $\lambda(a)$ and $\|a - \lambda(a)\|$ for particular a. These examples are used in later proofs. An expression for $\|\mathrm{ad}_A\, a\|$ when A is primitive is given (Theorem 3.8) and an important formula for $d(a, Z(A))$ is found (Theorem 3.11).

In the fourth chapter the pure functionals of A and a numerical range $U_A(a)$, introduced in [30], are discussed. Inequalities connecting $\lambda(a)$ and the circumcentre of $U_A(a)$ are proved (Lemma 4.5) and an important expression for $\|\mathrm{ad}_A\, a\|$ is given (Proposition 4.4).

In the fifth chapter the results built up in the previous sections are used to prove the main characterisation theorem (Theorem 5.7). We see that, for a unital C*-algebra A, the possible values of $K(A)$ and $K_s(A)$ fall into four cases:

(i) $K(A) = K_s(A) = 0$;

(ii) $K(A) = K_s(A) = \frac{1}{2}$;

(iii) $K(A) = \frac{1}{\sqrt{3}}$, $K_s(A) = \frac{1}{2}$;

(iv) $K(A) \geq K_s(A) \geq 1$.

The cases are characterised by certain intersection properties of the primitive ideals of A.

In the sixth chapter we present a brief review of other results on Archbold constants, which are of relevance to this paper.

Chapter 2

Preliminaries

In this chapter we present a brief review of some well known facts about representations of unital C*-algebras. For a more detailed presentation, see, for example, [19], [23] or [27]. We then introduce the spaces $\operatorname{Spec} A$, $\operatorname{Prim} A$ and $\operatorname{Glimm} A$ and some of their topologies.

The set of all closed two-sided ideals of a unital C*-algebra A is denoted by $\operatorname{Id} A$. We will follow convention and use the word ideal to mean a closed two-sided ideal. It can be shown that every element of $\operatorname{Id} A$ is a C*-subalgebra of A and contains an approximate unit, see [23], Theorems 3.1.1 and 3.1.3.

A **representation** of a unital C*-algebra A is a pair (π, H), where H is a Hilbert space and π is a *-homomorphism $\pi : A \to B(H)$. If π is injective then the representation is said to be **faithful**. If ξ is an element of H such that $\pi(A)\xi$ is dense in H, then the representation is said to be **cyclic** with **cyclic vector** ξ. Let $\pi(A)'$ be the **commutant** of $\pi(A)$, that is the set of elements of $B(H)$ which commute with every element of $\pi(A)$. Then (π, H) is said to be **irreducible** if $\pi(A)'$ is the set of scalar multiples of 1_H. The set of non-zero irreducible representations of A is denoted $\operatorname{Irr} A$.

The set $\pi(A)$ is the image of a *-homomorphism and $\pi(A)'$ is a norm-closed *-subalgebra. Hence both are C*-subalgebras of $B(H)$. If (π, H) is cyclic, $\pi(1)$ is the identity on $\pi(A)\xi$ and hence on H. As for any *-homomorphism, $\|\pi(a)\| \leq \|a\|$.

Let x be an element of the state space S_A of A. It is easy to check that the set

$$L_x = \{b \in A : x(b^*b) = 0\}$$

is a closed left ideal of A, and that, for all a and b in A,

$$\langle a + L_x, b + L_x \rangle = x(b^*a)$$

is a well-defined inner product on A/L_x. Let H_x be the completion of A/L_x and let $\pi_x(a)$ denote the unique extension of the well-defined map $b + L_x \to ab + L_x$ to H_x. Then $\pi_x(a)$ is an element of $B(H_x)$ and $\pi_x : a \to \pi_x(a)$ is a *-homomorphism. Thus (π_x, H) is a representation of A. Denote $1 + L_x$ by ξ_x. Then ξ_x is a cyclic vector of unit norm and x is recovered by the expression

$$x(a) = \langle \pi_x(a)\xi_x, \xi_x \rangle$$

for a in A. The representation (π_x, H_x) is called the **Gelfand-Naimark-Segal** or **GNS representation**.

Let (π, H) be a representation of A. For ξ in H, define ω_ξ in $B(H)^*$ by

$$\omega_\xi(T) = \langle T\xi, \xi \rangle$$

for T in $B(H)$. Recall that T is a positive element of a C*-subalgebra of $B(H)$ if and only if $\langle T\xi, \xi \rangle$ is positive for all ξ in H, [19], Theorem 4.2.6. Thus, $\omega_\xi|_{\pi(A)}$ is an element of $\pi(A)^*_+$. If ξ is of norm 1 and (π, H) is cyclic then ω_ξ lies in the unit ball, 1_H lies in $\pi(A)$ and $\omega_\xi(1_H)$ is 1. Hence, $\omega_\xi|_{\pi(A)}$ is a state of $\pi(A)$. Such an element of $S_{\pi(A)}$ is said to be a **vector state** of $\pi(A)$.

Let $\partial_e S_A$ be the set of pure states of A. The following proposition may be found in [19], Theorem 10.2.3.

Proposition 2.1. *Let x be an element of S_A. Then (π_x, H_x) is an element of* $\operatorname{Irr} A$ *if and only if x is an element of $\partial_e S_A$.*

Proof. Suppose that x is an element of $\partial_e S_A$. Let S be a positive element of the unit ball of $\pi_x(A)'$. Since (π_x, H_x) is cyclic and ξ_x has norm 1, it follows that $z = \omega_{\xi_x}|_{\pi_x(A)}$ is a vector state of $\pi_x(A)$. Let y be the element of $\pi_x(A)_+^*$ given by

$$y = \omega_{S^{\frac{1}{2}}\xi_x}|_{\pi_x(A)}.$$

For T in $\pi_x(A)_+$, the commutivity of $S^{\frac{1}{2}}$ and $T^{\frac{1}{2}}$ implies that:

$$y(T) = \left\|S^{\frac{1}{2}}T^{\frac{1}{2}}\xi_x\right\| \leq \left\|S^{\frac{1}{2}}\right\|\left\|T^{\frac{1}{2}}\xi_x\right\| \leq z(T)$$

Thus y and $z - y$ lie in $\pi_x(A)_+^*$, and it follows that there exist states x_1 and x_2 of $\pi_x(A)$, and positive real numbers a_1 and a_2, such that

$$y = a_1x_1, \qquad z - y = a_2x_2.$$

Since 1_H lies in $\pi_x(A)$, we may evaluate $z = a_1x_1 + a_2x_2$ at 1_H to find that $a_1 + a_2 = 1$. Furthermore,

$$z \circ \pi_x(a) = \langle \pi_x(a)\xi_x, \xi_x\rangle = x(a),$$

and

$$x = a_1(x_1 \circ \pi_x) + a_2(x_2 \circ \pi_x).$$

We also have that $x_i \circ \pi_x$ lies in S_A, because $\|x_i \circ \pi_x\| \leq 1$ and

$$x_i \circ \pi_x(1) = x_i(1_H) = 1.$$

Since x is pure it follows that $y = a_1 z$. For P and Q in $\pi_x(A)$,

$$\begin{aligned}\langle SP\xi_x, Q\xi_x\rangle &= \left\langle Q^*PS^{\frac{1}{2}}\xi_x, S^{\frac{1}{2}}\xi_x\right\rangle \\ &= y(Q^*P) \\ &= a_1 z(Q^*P) \\ &= a_1\langle P\xi_x, Q\xi_x\rangle\end{aligned}$$

Hence S coincides with $a_1 1_{H_x}$ on $\pi(A)\xi_x$. Since $\pi(A)\xi_x$ is dense in H_x we have that S is a scalar multiple of 1_H. Finally, the positive elements of the unit ball of $\pi_x(A)'$ linearly generate $\pi_x(A)'$, which implies that $\pi_x(A)'$ is $\mathbb{C}1_H$.

Conversely, suppose that (π_x, H_x) is irreducible. Assume that

$$x = tx_1 + (1-t)x_2$$

for some x_1 and x_2 in S_A and t in $(0,1)$. Define $\sigma : \pi(A)\xi_x \times \pi(A)\xi_x \to \mathbb{C}$ by

$$\sigma(\pi_x(a)\xi_x, \pi_x(b)\xi_x) = tx_1(b^*a)$$

for a and b in A. Since

$$\begin{aligned}|tx_1(b^*a)|^2 &\leq tx_1(a^*a)\,tx_1(b^*b) \\ &\leq x(a^*a)\,x(b^*b) \\ &= \|\pi_x(a)\xi_x\|^2\|\pi_x(b)\xi_x\|^2\end{aligned}$$

we see that σ is well defined and

$$|\sigma(\xi,\eta)| \leq \|\xi\|\|\eta\|$$

for all ξ and η in $\pi_x(A)\xi_x$. Thus σ is continuous in each variable. It is also sesquilinear. Then for ξ and η in $\pi_x(A)\xi_x$, the map $\xi \to \sigma(\xi,\eta)$ is a linear functional. Extending to H_x and applying the Riesz representation theorem shows that, for each η in $\pi_x(A)\xi_x$, there exists a unique ζ_η in H_x such that $\|\zeta_\eta\| \le \|\eta\|$ and

$$\sigma(\xi,\eta) = \langle \xi, \zeta_\eta \rangle$$

for all ξ in $\pi_x(A)\xi_x$. Define $T : \pi_x(A)\xi_x \to H_x$ by

$$T\eta = \zeta_\eta$$

for η in $\pi_x(A)\xi_x$. Then T is bounded and

$$\sigma(\xi,\eta) = \langle \xi, T\eta \rangle$$

for all ξ and η in $\pi_x(A)\xi_x$. By the uniqueness of ζ_η we see that T is linear on $\pi_x(A)\xi_x$. Thus, we may extend T to an element of $B(H)$. Since x_1 lies in A_+^*, it is self-adjoint. Thus, for a and b in A,

$$\begin{aligned}\overline{\sigma(\pi_x(b)\xi_x, \pi_x(a)\xi_x)} &= t\overline{x_1(a^*b)} \\ &= tx_1(b^*a) \\ &= \sigma(\pi_x(a)\xi_x, \pi_x(b)\xi_x).\end{aligned}$$

Hence, for all ξ and η in $\pi_x(A)\xi_x$,

$$\langle T^*\xi, \eta \rangle = \langle \xi, T\eta \rangle = \langle T\xi, \eta \rangle.$$

By continuity this expression holds for all η in H_x and it follows that T is self-adjoint on $\pi_x(A)\xi_x$ and hence on H_x. Let a, b and c be elements of A. Then

$$\begin{aligned}\langle (\pi_x(a)T - T\pi_x(a))\pi_x(b)\xi_x, \pi_x(c)\xi_x \rangle &= \langle \pi_x(b)\xi_x, T\pi_x(a^*c)\xi_x \rangle \\ &\quad - \langle \pi_x(ab)\xi_x, T\pi_x(c)\xi_x \rangle \\ &= tx_1(c^*ab) - tx_1(c^*ab) \\ &= 0.\end{aligned}$$

Therefore, using continuity again,

$$\langle (\pi_x(a)T - T\pi_x(a))\xi, \eta \rangle = 0$$

for ξ in $\pi_x(A)\xi_x$ and η in H_x. It follows that, for all a in A,

$$\pi_x(a)T = T\pi_x(a).$$

Thus T is in $\pi(A)'_{sa}$ and, by hypothesis, $T = \lambda 1_{H_x}$ for some λ in $\mathbb{R}$. Thus

$$tx_1(b^*a) = \lambda x(b^*a).$$

When b is equal to 1

$$tx_1(a) = \lambda x(a)$$

for all a in A. Evaluating when a is 1 then shows that λ equals t so that x_1 agrees with x. Thus x is pure. □

Proposition 2.1 shows that there exists a map $\Xi : \partial_e S_A \to \operatorname{Irr} A$ given by $x \to (\pi_x, H_x)$.

Let H_1, H_2 be Hilbert spaces. An element U in $B(H_1, H_2)$ is said to be unitary if U is a surjective isometry. Representations (π_1, H_1) and (π_2, H_2) are said to be unitarily equivalent if there exists a unitary U in $B(H_1, H_2)$ such that, for all a in A,

$$\pi_2(a)U = U\pi_1(a).$$

Unitary equivalence is clearly a reflexive and transitive relation. To see that it is symmetric, note that $\|U\xi_1\| = \|\xi_1\|$ implies that $\langle \xi_1, (U^*U - I)\,\xi_1\rangle$ is zero for all ξ_1 in H_1. By [22], Lemma 3.9-3 (b), U^*U is the identity, and, with surjectivity of U, this implies that UU^* is also the identity. Hence U^* is unitary, and, for all ξ_1 in H_1 and ξ_2 in H_2,

$$\begin{aligned}\langle \pi_1(a)\,U^*\xi_2, \xi_1\rangle &= \langle \xi_2, U\pi_1(a^*)\,\xi_1\rangle \\ &= \langle \xi_2, \pi_2(a^*)\,U\xi_1\rangle \\ &= \langle U^*\pi_2(a)\,\xi_2, \xi_1\rangle .\end{aligned}$$

Therefore $\pi_1(a)\,U^*$ equals $U^*\pi_2(a)$ for all a in A.

Define the **spectrum** of A, denoted $\operatorname{Spec} A$ or $\hat{A}$, to be the set of unitary equivalence classes of $\operatorname{Irr} A$. Then there exists a mapping $p_{\approx} \circ \Xi : \partial_e S_A \to \operatorname{Spec} A$ where $p_{\approx}$ is the quotient map. We will now show that $p_{\approx} \circ \Xi$ is surjective.

Lemma 2.2. *Let (π, H) be an irreducible representation. Let V be a closed subspace of H, invariant under π. Then V is zero or H.*

Proof. Since V is a closed subspace, $H = V \oplus V^{\perp}$. Let p be the natural projection onto V. For any η in $V^{\perp}$ we have

$$\langle \pi(a)\,\eta, \xi\rangle = \langle \eta, \pi(a^*)\,\xi\rangle = 0$$

for all a in A and hence $V^{\perp}$ is π-invariant. Since

$$\begin{aligned}p\pi(a)\,(\xi_1 + \xi_2) &= \pi(a)\,\xi_1 \\ &= \pi(a)\,p\,(\xi_1 + \xi_2)\end{aligned}$$

for all a in A, ξ_1 in V and ξ_2 in $V^{\perp}$, p lies in $\pi(a)'$. This implies that

$$p = p^2 = \lambda 1_H.$$

Thus, $\lambda(\lambda - 1)$ is zero and p is zero or the identity. Therefore V is zero or H since p coincides with the identity on V. ☐

Corollary 2.3. *Let (π, H) be an element of* $\operatorname{Irr} A$. *Then every non-zero element of H is a cyclic vector.*

Proof. Let ξ be an element of H. Then $\overline{\pi(A)\,\xi}$ is a subspace of H. If $(\pi(a_n)\,\xi)$ is a sequence, norm-convergent to η in $\overline{\pi(A)\,\xi}$, then

$$\|\pi(a)\,\eta - \pi(aa_n)\,\xi\| \leq \|\pi(a)\|\,\|\eta - \pi(a_n)\,\xi\|$$

and the right-hand side converges to zero. Hence $\overline{\pi(A)\,\xi}$ is π-invariant and $\overline{\pi(A)\,\xi}$ is zero or H. By linearity and continuity of each $\pi(a)$

$$N = \bigcap_{a \in A} \{\xi \in H : \pi(a)\,\xi = 0\}$$

is a closed vector subspace . Since $\pi(a)\,\xi$ is zero for all a in A and ξ in N, and zero lies in N, N is π-invariant. Then N is zero as π is non-zero. Finally, $\pi(A)\,\xi$ is zero if and only if $\overline{\pi(A)\,\xi}$ is zero, if and only if ξ is zero, so every non-zero ξ in H is cyclic. ☐

Theorem 2.4. *For each element (π, H) in* $\operatorname{Irr} A$, *there exists an element x of $\partial_e S_A$ such that (π, H) is unitarily equivalent to (π_x, H_x).*

Proof. Since H is non-trivial it has a vector η of unit norm. Let $x = \omega_\eta \circ \pi$ be a state of A. Define $U : \pi(A)\,\eta \to \pi_x(A)\,\xi_x$ by

$$U\pi(a)\,\eta = \pi_x(a)\,\xi_x$$

for a in A. Then U is linear and surjective. Furthermore, for all a in A,

$$\begin{aligned}\|U\pi(a)\eta\|^2 &= \|\pi_x(a)\xi_x\|^2\\ &= \langle \pi_x(a^*a)\xi_x, \xi_x\rangle\\ &= x(a^*a)\\ &= \omega_\eta\circ\pi(a^*a)\\ &= \|\pi(a)\eta\|^2.\end{aligned}$$

Therefore U is isometric. Since, for all a and b in A,

$$U\pi(a)\pi(b)\eta = \pi_x(a)\pi_x(b)\xi_x = \pi_x(a)U\pi(b)\eta,$$

$U\pi(a)$ equals $\pi_x(a)U$. Extending U to H_1 preserves these properties. Since (π, H) is irreducible, (π_x, H_x) is irreducible, and, by Proposition 2.1, x is pure. □

Hence we see that the mapping $p_{\approx}\circ\Xi$ is surjective.

If (π_1, H_1) and (π_2, H_2) are unitarily equivalent and a is an element of $\ker\pi_1$ then

$$\pi_2(a)\xi_2 = \pi_2(a)U\xi_1 = U\pi_1(a)\xi_1 = 0$$

which implies that a is also an element of $\ker\pi_2$. By symmetry, $\ker\pi_1$ equals $\ker\pi_2$. Since π is a *-homomorphism, $\ker\pi$ is an ideal. Thus there is a well-defined map $\ker : \operatorname{Spec} A \to \operatorname{Id} A$. Define the **primitive spectrum** of A denoted $\operatorname{Prim} A$ or $\check{A}$ to be the range of ker. Elements of $\operatorname{Prim} A$ are said to be **primitive ideals** and clearly the mapping $\ker\circ p_{Irr}\circ\Xi$ is surjective. A C*-algebra A is said to be **primitive** if $\{0\}$ is a primitive ideal. Thus, A is primitive if and only if it has a faithful irreducible representation. Equivalently, primitive C*-algebras are isometrically *-isomorphic to C*-subalgebras of $B(H)$ with commutant $\mathbb{C}1_H$, for some Hilbert space H.

Let $\ker\pi_x$ be an element of $\operatorname{Prim} A$, where x is an element of $\partial_e S_A$. Then $\pi_x(a)H_x$ is zero if and only if $\pi_x(a)A/L_x$ is zero, if and only if aA is a subset of L_x. Therefore,

$$\ker\pi_x = \{a\in A : aA\subseteq L_x\}.$$

Since A is unital, this implies that $\ker\pi_x$ is a subset of L_x. If I is an element of $\operatorname{Id} A$ and I is a subset of L_x then

$$IA\subseteq I\subseteq L_x,$$

and hence I is a subset of $\ker\pi_x$.

An ideal I of A is said to be **prime** if, whenever I_1 and I_2 are ideals of A such that I_1I_2 is a subset of I, then at least one of I_1 and I_2 is a subset of I. The ideal I is said to be **primal** if, whenever I_1 and I_2 are such that I_1I_2 is zero, at least one of I_1 and I_2 is a subset of I. The set of all primal ideals of A is denoted $\operatorname{Primal} A$. Clearly A is always prime and every prime ideal is primal.

Suppose that $I_1I_2\subseteq\ker\pi_x\subseteq L_x$ and that I_2 is not a subset of L_x. Then $\pi_x(I_2)\xi_x$ is $\pi_x(a)$-invariant, (π_x, H_x) is an element of $\operatorname{Irr} A$ and $\pi_x(I_2)\xi_x$ is non-zero. Hence $\pi_x(I_2)\xi_x$ is H_x by Lemma 2.2 and ξ_x lies in $\pi_x(I_2)\xi_x$. Since I_1I_2 is a subset of L_x it follows that

$$\pi_x(I_1)\xi_x\subseteq\pi_x(I_1)\pi_x(I_2)\xi_x = \pi_x(I_1I_2)\xi_x = \{0\}.$$

Therefore $\pi_x(I_1)\xi_x$ is zero and I_1 is a subset of L_x. Thus, if I_1I_2 is a subset of $\ker\pi_x$, at least one of I_1 and I_2 is a subset of L_x, and hence of $\ker\pi_x$. We conclude that every primitive ideal is prime.

Let a be an element of $(I_1\cap I_2)^+$ and let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for I_1. Then

$$a = \lim_\lambda\left(u_\lambda a^{\frac{1}{2}}\right)a^{\frac{1}{2}},$$

where $u_\lambda a^{\frac{1}{2}}$ lies in I_1 and $a^{\frac{1}{2}}$ lies in I_2. Hence a is an element of I_1I_2. This extends linearly to $I_1\cap I_2$, implying that $I_1\cap I_2$ is a subset of I_1I_2. Clearly $I_1\cap I_2$ is also a superset of I_1I_2 and I_1I_2 equals $I_1\cap I_2$ for all I_1 and I_2 in $\operatorname{Id} A$.

If S is a subset of A, define the hull of S, denoted $\operatorname{hull} S$, to be the set of primitive ideals of A containing S. If X is a subset of $\operatorname{Prim} A$, define the kernel of X, denoted $\ker X$, to be the intersection of the ideals in X. By convention $\ker\varnothing$ is A. Define $\overline{X}$ to be the set $\operatorname{hull}\ker X$.

Proposition 2.5. *The map taking X to $\bar{X}$ satisfies the Kuratowski closure axioms:*

(i) $\overline{\emptyset} = \emptyset$;

(ii) $X \subseteq \overline{X}$;

(iii) $\overline{\overline{X}} = \overline{X}$;

(iv) $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$;

for all subsets X and Y of $\mathrm{Prim}\, A$. *Hence the set*

$$\tau_J = \left\{\mathrm{Prim}\, A \backslash \overline{X} : X \subseteq \mathrm{Prim}\, A\right\}$$

is a topology for $\mathrm{Prim}\, A$.

Proof. The proof is as follows:

(i) $\overline{\emptyset} = \mathrm{hull}\ker\emptyset = \mathrm{hull}\, A = \emptyset$.

(ii) Let P be an element of X. Then $\ker X$ is a subset of P which implies that P lies in $\mathrm{hull}\ker X$. But $\mathrm{hull}\ker X$ is $\overline{X}$. Therefore X is a subset of $\overline{X}$.

(iii) Let P be an element of $\overline{\overline{X}}$. Then P contains $\ker\mathrm{hull}\ker X$ and hence $\ker X$. Thus P is an element of $\mathrm{hull}\ker X$ which is $\overline{X}$. Thus $\overline{\overline{X}}$ is a subset of $\overline{X}$ and by (2), $\overline{X}$ is a subset of $\overline{\overline{X}}$.

(iv) Let P be an element of $\overline{X}$. Then P contains $\ker X \cup Y$ and therefore P lies in $\overline{X \cup Y}$. Hence $\overline{X}$ is a subset of $\overline{X \cup Y}$. Similarly for Y, so $\overline{X} \cup \overline{Y}$ is a subset of $\overline{X \cup Y}$. Conversely, let P be an element of $\overline{X \cup Y}$. Then

$$P \supseteq \ker X \cup Y = \ker X \cap \ker Y = \ker X \ker Y$$

Therefore P contains one of $\ker X$ and $\ker Y$ since P is prime. Thus P is an element of $\overline{X}$ or $\overline{Y}$ so $\overline{X \cup Y}$ is a subset of $\overline{X} \cup \overline{Y}$.

□

The topology τ_J is the **Jacobson** or **hull-kernel** topology for $\mathrm{Prim}\, A$.
.

Proposition 2.6. *Let I be an ideal of a unital C*-algebra. Then*

$$I = \ker\mathrm{hull}\, I.$$

Proof. The result is immediate if I is A. Consider I to be a proper ideal. It is obvious that I is a subset of $\ker\mathrm{hull}\, I$. If a lies in $A \backslash I$ then a_I is non-zero and there exists an element x in $\partial_e S_{A/I}$ such that $x\,(a_I)$ is non-zero ([19], Theorem 4.3.8). Then $\langle \pi_x\,(a_I)\,\xi_x, \xi_x\rangle$ is non-zero, which implies that $\pi_x\,(a_I)$ is non-zero. Let $\pi = \pi_x \circ p_I$ where p_I is the quotient map. Then $\pi\,(A)$ agrees with $\pi_x\,(A/I)$ which implies that (π, H_x) lies in $\mathrm{Irr}\, A$. Since $\pi\,(I)$ is zero, $\ker\pi$ lies in $\mathrm{hull}\, I$. Since $\pi\,(a)$ is non-zero, a lies in the complement of $\ker\pi$ and hence in the complement of $\ker\mathrm{hull}\, I$. Therefore $\ker\mathrm{hull}\, I \subseteq I$. □

In particular,

$$\ker\mathrm{Prim}\, A = \ker\mathrm{hull}\,\{0\} = \{0\}$$

so the kernel of $\mathrm{Prim}\, A$ is zero. We also note that, for I proper, $\mathrm{hull}\, I$ is non-empty.

We now give a characterisation of primal ideals from [5], Proposition 3.2.

Theorem 2.7. *Let I be an ideal of a C*-algebra A. Then the following conditions are equivalent:*

(i) *I is a primal ideal of A;*

(ii) *whenever* $n \geq 1$ *and* $I_1, \ldots, I_n$ *are ideals of* A *such that* $I_j \nsubseteq I$ *for* $j = 1 \ldots n$ *then* $\prod_{j=1}^{n} I_j$ *is non-zero;*

(iii) *whenever* $n \geq 1$ *and* $U_1, \ldots, U_n$ *are open subsets of* $\operatorname{Prim} A$ *which intersect* $\operatorname{hull} I$ *then* $\bigcap_{j=1}^{n} U_j$ *is non-empty;*

(iv) *there is a net* $(P_\lambda)_{\lambda \in \Lambda}$ *in* $\operatorname{Prim} A$ *convergent to every point of* $\operatorname{hull} I$.

Proof. Equivalence of (1) and (2) is immediate from the definitions. Equivalence of (2) and (3) follows from identifying $\operatorname{Id} A$ with the open subsets of $\operatorname{Prim} A$ via the bijection $I \to (\operatorname{hull} I)^c$. Suppose that (1)-(3) hold. If P is a primitive ideal of A then we can choose an open neighbourhood U_P of P in $\operatorname{Prim} A$ (e.g. take U_P to be $\operatorname{Prim} A$). Let Λ be the set of indexed sets $(U_P)_{P \in \operatorname{hull} I}$ such that U_P is proper for only finitely many P in $\operatorname{hull} I$. Define a direction on Λ by

$$(U_P)_{P \in \operatorname{hull} I} \geq (V_P)_{P \in \operatorname{hull} I} \quad \Leftrightarrow U_P \subseteq V_P \ \forall P \in \operatorname{hull} I.$$

Let $(U_P)_{P \in \operatorname{hull} I}$ be an element of Λ. Let $P_1, \ldots P_n$ be the elements of $\operatorname{hull} I$ such that U_{P_j} are proper. Then P_j lies in $U_{P_j} \cap \operatorname{hull} I$ for $j = 1 \ldots n$. By (3) we can choose a primitive ideal P_λ in $\bigcap_{j=1}^{n} U_{P_j}$.

Let Q be an arbitrary element of $\operatorname{hull} I$ and let V_Q be an open neighbourhood of Q. Extend this to an element λ_0 of Λ by defining V_P to be $\operatorname{Prim} A$ for P not equal to Q. When $\lambda \geq \lambda_0$ we have that P_λ lies in U_Q which is a subset of V_Q. Hence $(P_\lambda)_{\lambda \in \Lambda}$ converges to Q and (4) holds.

If (4) holds and $U_1, \ldots, U_n$ are open subsets of $\operatorname{Prim} A$ which intersect $\operatorname{hull} I$ then there exists λ_0 in Λ such that P_λ lies in each U_j for $\lambda \geq \lambda_0$. Thus (3) holds. □

The following corollary will prove useful.

Corollary 2.8. *Let* $P_1, \ldots, P_n$ *be elements of* $\operatorname{Prim} A$ *and let* I *be their intersection. Then the following conditions are equivalent:*

(i) I *is a primal ideal of* A*;*

(ii) *when* $I_1, \ldots, I_n$ *are ideals of* A *such that* $I_j \nsubseteq P_j$ *for* $j = 1 \ldots n$ *then* $\prod_{j=1}^{n} I_j$ *is non-zero;*

(iii) *if* $U_1, \ldots, U_n$ *are open neighbourhoods of* $P_1, \ldots, P_n$ *respectively in* $\operatorname{Prim} A$ *then* $\bigcap_{j=1}^{n} U_j$ *is non-empty.*

Proof. The equivalence of (2) and (3) follows from the identification of $\operatorname{Id} A$ with the open subsets of $\operatorname{Prim} A$ as above. Suppose that (1) holds, and let $U_1, \ldots, U_n$ be open neighbourhoods of $P_1, \ldots, P_n$ in $\operatorname{Prim} A$. Since each P_j is in $\operatorname{hull} I$, (3) of Theorem 2.7 holds. Hence $\bigcap_{j=1}^{n} U_j$ is non-empty and (3) holds.

Conversely, suppose that (3) holds. Let $U_1, \ldots, U_m$ be open subsets of $\operatorname{Prim} A$ which intersect $\operatorname{hull} I$. Observe that

$$\operatorname{hull} I = \operatorname{hull} \ker \{P_1, \ldots, P_n\}$$

and that $\operatorname{hull} I$ is the closure of $\{P_1, \ldots, P_n\}$ in the Jacobson topology. Thus every U_j contains an element of $\{P_1, \ldots, P_n\}$. Define $V_1, \ldots, V_n$ by

$$V_k = \ker \{U_j : P_k \in U_j, 1 \leq j \leq n\}.$$

Then each V_k is an open neighbourhood of P_k and hence $\bigcap_{k=1}^{n} V_k$ is non-empty. But then $\bigcap_{j=1}^{n} U_j$ is non-empty so (2) of Theorem 2.7 holds and thus (1) holds, as required. □

As an immediate consequence of this corollary, we can greatly simplify the proof of [4], Proposition 4.5. A **minimal primal ideal** is a primal ideal which has no primal proper subsets. An element P in $\operatorname{Prim} A$ is **separated** if, for Q in the complement of $\operatorname{hull} P$, P and Q can be separated by disjoint open sets. In [4] the following proposition is proved using nets and facts about two topologies on $\operatorname{Primal} A$. The proof given here, which I believe is new, is much more elementary.

Proposition 2.9. *Let A be a C^*-algebra and let P be an element of* $\operatorname{Prim} A$*. Then the following conditions are equivalent:*

(i) *P is a minimal primal ideal of A;*

(ii) *P is a separated point in* $\operatorname{Prim} A$*.*

Proof. Suppose that P is a minimal primal ideal, but that P is not separated. Then there exists a primitive ideal Q not in $\operatorname{hull} P$ such that P and Q cannot be separated by disjoint open sets. So $P \cap Q$ is primal and a proper subset of P, which leads to a contradiction. Therefore (1) implies (2).

Conversely, suppose that P is separated but is not a minimal primal ideal. Then there exists a proper ideal I of P which is primal. Then $\operatorname{hull} P$ is a proper subset of $\operatorname{hull} I$, and there exists Q in $\operatorname{hull} I$ but not in $\operatorname{hull} P$. By hypothesis, P and Q can be separated by disjoint open sets, so $P \cap Q$ is not primal. But $P \cap Q$ contains a primal ideal, I, and is therefore primal, which leads to a contradiction. Therefore (2) implies (1). □

Proposition 2.10. *Let A be a unital C^*-algebra. Then* $\operatorname{Prim} A$ *is compact in the Jacobson topology.*

Proof. Let $\mathcal{U}$ be an open cover for $\operatorname{Prim} A$ and suppose that $\mathcal{U}$ has no finite subcover. Let $\mathcal{C}$ be the collection of closed sets $\{\operatorname{Prim} A \backslash U : U \in \mathcal{U}\}$. Let $C_1, \ldots, C_n$ be elements of $\mathcal{C}$ and let U_j be the complement of C_j in $\operatorname{Prim} A$. The intersection of $C_1, \ldots, C_n$ is non-empty, as otherwise $\{U_1, \ldots, U_n\}$ would be an open subcover for $\operatorname{Prim} A$. Let I be the set $\sum_{C \in \mathcal{C}} \ker C$. We now prove by contradiction that I is proper. Assume that I is A. Then there exist $a_1, \ldots, a_n$ in A and $C_1, \ldots, C_n$ in $\mathcal{C}$ such that a_j lies in $\ker C_j$ and

$$1 = a_1 + \ldots + a_n$$

This implies that $\ker C_1 + \ldots \ker C_n$ is A. Since $C_1, \ldots, C_n$ have non-empty intersection they have a common primitive ideal P. Since each C_j equals $\operatorname{hull} \ker C_j$ we have

$$P \supseteq \ker C_1 + \ldots \ker C_n = A,$$

which is impossible as primitive ideals are proper. Therefore I is a proper ideal of A and hence is contained in a primitive ideal Q. Then Q contains $\ker C$ for all C in $\mathcal{C}$ and is therefore an element of each C. Therefore the intersection of $\mathcal{C}$ is non-empty. But this contradicts the definition of $\mathcal{U}$ as an open cover for $\operatorname{Prim} A$, so by contradiction $\operatorname{Prim} A$ is compact. □

It is easy to check that $\left\{\ker^{-1} U : U \in \tau_J\right\}$ is a topology on $\operatorname{Spec} A$. This topology is called the **Jacobson** or **hull-kernel** topology for $\operatorname{Spec} A$. By construction the mapping $\ker : \operatorname{Spec} A \to \operatorname{Prim} A$ is continuous with respect to the Jacobson topologies and, since it is surjective, it is also open.

Let $\theta : \partial_e S_A \to \operatorname{Spec} A$ be the natural map defined by

$$\theta(x) = [(\pi_x, H_x)]$$

for x in $\partial_e S_A$. Then θ is surjective by Theorem 2.4 and every element of $\operatorname{Spec} A$ can be written $\theta(x)$ for some x in $\partial_e S_A$. We quote the following Lemma, which is the equivalence of (i) and (iv) in [10], 3.4.10.

Lemma 2.11. *Let $\partial_e S_A$ have the weak*-topology and let* $\operatorname{Spec} A$ *have the Jacobson topology. Let V be a subset of* $\operatorname{Spec} A$ *and let U be the set $\theta^{-1}(V)$. Then for x in $\partial_e S_A$ and $\theta(x)$ in* $\operatorname{Spec} A$*, $\theta(x)$ lies in $\overline{V}$ if and only if x lies in $\overline{U}$.*

With the notation of Lemma 2.11, suppose that V is closed and let x be an element of $\overline{U}$. Then by Lemma 2.11 we have that $\theta(x)$ is an element of V and hence that x is an element of U. Thus U is closed. Conversely, suppose that U is closed and let $\theta(x)$ be an element of $\overline{V}$. By Lemma 2.11 again, x lies in U. Then $\theta(x)$ lies in V which implies that V is closed. Thus we have shown that a subset V of $\operatorname{Spec} A$ is closed if and only if $\theta^{-1}(V)$ is closed. Therefore θ is continuous.

Now let U be open in $\partial_e S_A$ and let V be the set $\theta(U)$. Suppose that $\theta(x)$ lies in $\overline{\operatorname{Spec} A \backslash V}$ for some x in U. Then Lemma 2.11 implies that x lies in $\overline{\theta^{-1}(\operatorname{Spec} A \backslash V)}$. Since $\theta^{-1}(\operatorname{Spec} A \backslash V)$ is a subset of $\partial_e S_A \backslash U$, this implies that x lies in $\partial_e S_A \backslash U$, which is a contradiction. Thus $\operatorname{Spec} A \backslash V$ is closed and V is open so θ is an open map.

The following Theorem is the Second Dauns-Hofmann Theorem, see [9]:

Theorem 2.12. *Let A be a unital C*-algebra and let $C^b(\mathrm{Prim}\, A)$ be the ring of continuous bounded functions on $\mathrm{Prim}\, A$. Then for each a in A and each f in $C^b(\mathrm{Prim}\, A)$ there exists an element a^f of A such that for all P in $\mathrm{Prim}\, A$*

$$a_P^f = f(P)\, a_P.$$

The following corollary to the Second Dauns-Hofmann Theorem was also proved for the unital case in [9], Chapter III, Section 5, and for the non-unital case in [11]. Our proof follows [11], but with appropriate simplifications for the unital case.

Theorem 2.13. *Let A be a unital C*-algebra. Then there is a *-isomorphism $\gamma : Z(A) \to C^b(\mathrm{Prim}\, A)$ such that, for all P in $\mathrm{Prim}\, A$,*

$$h_z(P)\, 1_{H_x} = \pi_x(z),$$

where

$$h_z = \gamma(z)$$

for each z in $Z(A)$, and where x in $\partial_e S_A$ is such that P is the kernel of π_x.

Proof. Let z be an element of $Z(A)$. Define $f_z : \partial_e S_A \to \mathbb{C}$ by

$$f_z(x) = x(z)$$

for x in $\partial_e S_A$. Since (π_x, H_x) is irreducible, $\pi_x(z)$ agrees with $\lambda 1_{H_x}$ for some complex number λ. In fact

$$f_z(x) = x(z) = \langle \pi_x(z)\, \xi_x, \xi_x \rangle = \lambda.$$

Hence,

$$\pi_x(z) = f_z(x)\, 1_{H_x}.$$

If x and y are elements of $\partial_e S_A$ such that (π_x, H_x) and (π_y, H_y) are unitarily equivalent, then there exists a unitary U in $B(H_x, H_y)$ such that, for all a in A,

$$U \pi_x(a) = \pi_y(a)\, U.$$

By [24], Proposition 3.13.4, there exists a u in $\mathcal{U}(A)$ such that

$$U \xi_x = \pi_y(u)\, \xi_y.$$

Therefore

$$\begin{aligned} x(a) &= \langle \pi_x(a)\, \xi_x, \xi_x \rangle \\ &= \langle U \pi_x(a)\, \xi_x, U \xi_x \rangle \\ &= \langle \pi_y(a)\, \pi_y(u)\, \xi_y, \pi_y(u)\, \xi_y \rangle \\ &= y(u^* a u). \end{aligned}$$

In particular,

$$f_z(x) = x(z) = y(z) = f_z(y).$$

Hence the map $g_z : \mathrm{Spec}\, A \to \mathbb{C}$ given by

$$g_z([(\pi_x, H_x)]) = f_z(x)$$

is well defined. Clearly f_z agrees with $g_z \circ \theta$. Since f_z is $\hat{z}$, f_z is continuous on $\partial_e S_A$ with the weak*-topology and since θ is open, g_z is continuous.

Let x and y be elements of $\partial_e S_A$ such that π_x and π_y have the same kernel. Then $\pi_x(z - f_z(x))$ is zero and hence $\pi_y(z - f_z(x))$ is zero. Therefore

$$f_z(y)\, 1_{H_y} - f_z(x)\, 1_{H_y} = 0$$

and $f_y(y)$ equals $f_z(x)$. Hence there is a well defined function $h_z : \operatorname{Prim} A \to \mathbb{C}$ given by

$$h_z(\ker \pi_x) = f_z(x)$$

for x in $\partial_e S_A$. Since g_z agrees with $h_z \circ \ker$, the map ker is open and g_z is continuous, it follows that h_z is continuous. Furthermore,

$$|h_z(\ker \pi_x)| = |x(z)| \leq \|z\|$$

so h_z is a continuous bounded function on $\operatorname{Prim} A$. Thus a map $\gamma : Z(A) \to C^b(\operatorname{Prim} A)$ may be defined by $\gamma(z) = h_z$ for z in $Z(A)$. If P is an element of $\operatorname{Prim} A$, then for x in $\partial_e S_A$ such that P is the kernel of π_x, it follows that

$$h_z(P) 1_{H_x} = f_z(x) 1_{H_x} = \pi_x(z).$$

Since π_x is a *-homomorphism it is easy to check that γ is a *-homomorphism.

If $\gamma(z_1)$ equals $\gamma(z_2)$ then $\widehat{z_1}$ agrees with $\widehat{z_2}$ on $\partial_e S_A$ so $x(z_1 - z_2)$ is zero for all x in $\partial_e S_A$. Thus z_1 equals z_2 by [19], paragraph 4.3.8. Hence γ is injective.

Let h be an element of $C^b(\operatorname{Prim} A)$. By the Second Dauns-Hofmann Theorem there exists z in A such that

$$z_P = h(P) 1_P$$

for all P in $\operatorname{Prim} A$. For a in A and P in $\operatorname{Prim} A$,

$$(az - za)_P = h(P) a_p 1_p - h(P) 1_p a_p = 0_P$$

so $az - za$ lies in the kernel of $\operatorname{Prim} A$ and is therefore zero. This implies that z lies in $Z(A)$. Since

$$h_z(P) 1_{H_x} = \pi_x(z)$$

when P is the kernel of π_x for some x in $\partial_e S_A$, it follows that $h_z(P) - z$ lies in P. Thus

$$h_z(P) 1_P = z_P = h(P) 1_P$$

for all P in $\operatorname{Prim} A$ so $h = \gamma(z)$. Thus γ is surjective, which completes the proof. □

Let A be a C*-algebra and let $\Delta(A)$ be the set of characters of A.

Theorem 2.14. *Let A be a unital C*-algebra. Then $\Delta(A)$ is a subset of $\partial_e S_A$. When A is commutative $\Delta(A)$ and $\partial_e S_A$ coincide.*

Proof. Let x be an element of $\Delta(A)$. Since x is non-zero there exists a in A such that $x(a)$ is non-zero. Then $x(a)(x(1) - 1)$ is zero and, as for any *-homomorphism, x lies in A_1^* so $x(1)$ and $\|x\|$ are 1. Hence x is a state. Since x is a *-homomorphism, L_x is the kernel of x. For any a in A, $a - x(a) 1$ is in the kernel of x so A/L_x is isometrically *-isomorphic to $\mathbb{C}$, as are H_x and $B(H_x)$. Thus (π_x, H_x) is irreducible,

$$\pi_x(a) = x(a) 1_{H_x}$$

for all a in A, and, by Theorem 2.4, there exists y in $\partial_e S_A$ and a unitary $U : H_x \to H_y$ such that, for all a in A,

$$\pi_y(a) U = U \pi_x(a)$$

Now, for all a in A,

$$\begin{aligned} y(a) &= \langle \pi_y(a) U U^* \xi_y, \xi_y \rangle \\ &= \langle U \pi_x(a) U^* \xi_y, \xi_y \rangle \\ &= x(a) \langle U U^* \xi_y, \xi_y \rangle \\ &= x(a) \end{aligned}$$

Thus x is a pure state.

Now suppose that A is commutative and let x be a pure state. Since A is commutative, $\pi_x(A)$ is a subset of $\pi_x(A)'$ and hence of the set of scalar operators. Thus, for all a in A,

$$\pi_x(a) = x(a)\, 1_{H_x}$$

and since π_x is a non-zero *-homomorphism it follows that x is. Thus x lies in $\Delta(A)$. □

In particular we note that $\Delta(Z(A))$ is $\partial_e S_{Z(A)}$ for any unital C*-algebra A. For x in $\Delta(Z(A))$ we denote the proper ideal $\ker x$ by I_x and define K_x to be the norm closure of $I_x A$. From the proof above we have that I_x is also $\ker \pi_x$ and that $Z(A)$ is the direct sum of I_x and the scalars, from which it follows that I_x is a maximal ideal of $Z(A)$. Note that

$$\operatorname{Prim} Z(A) = \{I_x : x \in \Delta(Z(A))\}.$$

Let P be a primitive ideal of A. Then there exists y in $\partial_e S_A$ such that P is the kernel of π_y. Let x be the restriction of y to $Z(A)$. Then, arguing as in the proof of Theorem 2.14 above,

$$\pi_y(z) = x(z)\, 1_{H_y}$$

for all z in $Z(A)$. Hence x is a character of $Z(A)$ and $P \cap Z(A)$ coincides with I_x. We may therefore define a map $\theta : \operatorname{Prim} A \to \operatorname{Prim} Z(A)$ by

$$\theta(P) = P \cap Z(A)$$

for P in $\operatorname{Prim} A$. Let x be an element of $\Delta(Z(A))$. Then

$$F = \bigcap_{z \in Z(A)} \hat{z}^{-1}(\{x(z)\})$$

is a non-empty closed face of S_A. By the Krein-Milman Theorem, F has an extreme point $\hat{x}$ in $\partial_e S_A$ which extends x. Furthermore, $\theta(\ker \pi_y)$ equals I_x, so θ is surjective. Let C be a closed subset of $\operatorname{Prim} Z(A)$. Then C is the hull of an ideal I of $Z(A)$, with respect to $Z(A)$. Now

$$\theta^{-1}(C) = \{P \in \operatorname{Prim} A : P \cap Z(A) \supseteq I\} = \operatorname{hull} I.$$

Hence $\theta^{-1}(C)$ is closed in $\operatorname{Prim} A$ and θ is continuous.

Let P be a primitive ideal of A. By surjectivity of θ, there exists a character x of $Z(A)$ such that $P \cap Z(A)$ coincides with I_x. Clearly K_x lies in P. Conversely, if K_x lies in P for some x in $\Delta(Z(A))$, then $P \cap Z(A)$ contains I_x. In fact $P \cap Z(A)$ equals I_x since I_x is maximal. Therefore, if P is a primitive ideal and x is a character of $Z(A)$, then

$$P \cap Z(A) = I_x \Leftrightarrow P \in \operatorname{hull} K_x$$

Define a relation $\approx$ on $\operatorname{Prim} A$ by

$$P \approx Q \;\Leftrightarrow\; f(P) = f(Q)\;\; \forall\, f \in C^b(\operatorname{Prim} A)$$

where P and Q are primitive ideals. Clearly $\approx$ is an equivalence relation. Let $[P]$ denote the equivalence class of P in $\operatorname{Prim} A$ and let $\operatorname{Prim} A/\approx$ denote the set of equivalence classes.

Let x and y be elements of $\partial_e S_A$ and let P and Q be the kernels of π_x and π_y respectively. Suppose that $P \approx Q$, let z be an element of $P \cap Z(A)$ and let f be the element of $C^b(\operatorname{Prim} A)$ induced by z under the *-isomorphism of Theorem 2.13. Then,

$$f(P)\, 1_{H_x} = \pi_x(z) = 0,$$

which implies that $f(P)$ and $f(Q)$ are zero. Thus,

$$\pi_y(z) = f(Q)\, 1_{Hy} = 0,$$

which implies that z is an element of $Q \cap Z(A)$. Therefore $P \cap Z(A)$ is a subset of $Q \cap Z(A)$ and, by symmetry, the sets are equal.

Conversely, suppose that $P \cap Z(A)$ equals $Q \cap Z(A)$. Let f be an element of $C^b(\operatorname{Prim} A)$, denote $f(Q)$ by λ and define g to be the element $f - \lambda$ of $C^b(\operatorname{Prim} A)$. Let z be the element of $Z(A)$ which corresponds to g under the *-isomorphism of Theorem 2.13. Then

$$\pi_y(z) = g(Q)\, 1_{Hy} = 0$$

and z is an element of $P \cap Z(A)$, and hence of $Q \cap Z(A)$. This implies that

$$g(P)\, 1_{H_x} = \pi_x(z) = 0.$$

Therefore f has the same value at P and Q. Thus we have proved that for P and Q in $\operatorname{Prim} A$,

$$P \approx Q \;\Leftrightarrow\; P \cap Z(A) = Q \cap Z(A).$$

Therefore, for each P in $\operatorname{Prim} A$, there exists x in $\Delta(Z(A))$ such that

$$\begin{aligned}[P] &= \{Q \in \operatorname{Prim} A : Q \cap Z(A) = I_x\} \\ &= \operatorname{hull} K_x,\end{aligned}$$

and, by surjectivity of θ, given x in $\Delta(Z(A))$, there exists P in $\operatorname{Prim} A$ such that $[P]$ is $\operatorname{hull} K_x$. Thus,

$$\frac{\operatorname{Prim} A}{\approx} = \{\operatorname{hull} K_x : x \in \Delta(Z(A))\}.$$

In particular, $[P]$ is closed in the Jacobson topology for each P in $\operatorname{Prim} A$.

Define the **complete regularisation map** of $\operatorname{Prim} A$ to be the map $\phi : \operatorname{Prim} A \to \operatorname{Id} A$, given by

$$\phi(P) = \ker[P]$$

for P in $\operatorname{Prim} A$. The range of ϕ is denoted by $\operatorname{Glimm} A$ and its elements are referred to as **Glimm ideals**. This terminology arises because

$$\begin{aligned}\operatorname{Glimm} A &= \{\ker \operatorname{hull} K_x : x \in \Delta(Z(A))\} \\ &= \{K_x : x \in \Delta(Z(A))\}.\end{aligned}$$

Thus $\operatorname{Glimm} A$ is the set of ideals studied by Glimm in [15], Section 4.

Let G be a Glimm ideal and let Q be a primitive ideal such that ϕ maps Q to G. If G is a subset of a primitive ideal P then

$$P \in \operatorname{hull}\ker[Q] = \overline{[Q]} = [Q]$$

since $[Q]$ is closed. Hence $\phi(P)$ equals G. In particular, if $\ker[P]$ and $\ker[Q]$ agree, $[P]$ equals $[Q]$, so there is a bijection from $\operatorname{Prim} A/\approx$ to $\operatorname{Glimm} A$. The Jacobson topology on $\operatorname{Prim} A$ induces a quotient topology on $\operatorname{Prim} A/\approx$, and hence on $\operatorname{Glimm} A$. This is termed the quotient topology for $\operatorname{Glimm} A$. By construction, ϕ is continuous with respect to these topologies.

Let f be a continuous bounded function on $\operatorname{Prim} A$, and define a functional $\tilde{f}$ on $\operatorname{Prim} A/\approx$ by

$$\tilde{f}([P]) = f(P)$$

for P in $\operatorname{Prim} A$. It is straightforward to check that $\tilde{f}$ is a well-defined continuous function on $\operatorname{Prim} A/\approx$. Let $[P_1]$ and $[P_2]$ be distinct elements of $\operatorname{Prim} A/\approx$. Then there exists f in $C^b(\operatorname{Prim} A)$ such that $\tilde{f}([P_1])$ and $\tilde{f}([P_2])$ are distinct. Since $\mathbb{C}$ is Hausdorff, it contains disjoint open subsets V_1 and V_2, containing $\tilde{f}([P_1])$ and $\tilde{f}([P_2])$ respectively. Then $\tilde{f}^{-1}(V_1)$ and $\tilde{f}^{-1}(V_2)$ are disjoint open sets of $\operatorname{Prim} A/\approx$ containing $[P_1]$ and $[P_2]$ respectively. Hence $\operatorname{Glimm} A$ with the quotient topology is Hausdorff. Since A is unital, it follows from Proposition 2.10 that $\operatorname{Glimm} A$ is also compact in the quotient topology.

Let K_x be a Glimm ideal, where x is an element of $\Delta(Z(A))$. Then

$$\begin{aligned} K_x \cap Z(A) &= (\ker \operatorname{hull} K_x) \cap Z(A) \\ &= \bigcap_{P \in \operatorname{hull} K_x} (P \cap Z(A)) \\ &= I_x. \end{aligned}$$

Thus, there is a map $\psi : \operatorname{Glimm} A \to \operatorname{Prim} Z(A)$ defined by

$$\psi(G) = G \cap Z(A)$$

for G in $\operatorname{Glimm} A$. Equivalently,

$$\psi(K_x) = I_x,\ \psi(\ker[P]) = P \cap Z(A)$$

for x in $\Delta(Z(A))$ and P in $\operatorname{Prim} A$. Then $\psi \circ \phi$ agrees with θ, and, since θ is continuous, if U is open in $\operatorname{Prim} Z(A)$, then $\phi^{-1}\left(\psi^{-1}(U)\right)$ is open in $\operatorname{Prim} A$. Hence $\psi^{-1}(U)$ is open in $\operatorname{Glimm} A$, and it follows that ψ is continuous. Clearly ψ is surjective. If $\psi(\ker[P])$ equals $\psi(\ker[Q])$ for some P and Q in $\operatorname{Prim} A$ then $P \approx Q$ and hence ψ is injective. As shown in [9], Lemma 8.10, the Jacobson topology on $\operatorname{Prim} Z(A)$ agrees with the weak*-topology induced by $\Delta(Z(A))$, and is therefore Hausdorff. Thus, ψ is a continuous bijection from a compact space to a Hausdorff space, and, by [34], Theorem 5.9.1, ψ is a homeomorphism.

The following is a useful technical result which may be found in [12].

Lemma 2.15. *Let A be a C*-algebra, let X be a non-empty subset of* $\operatorname{Id} A$ *and let J be the intersection of X. Then, for all a in A,*

$$\|a_J\| = \sup\{\|a_I\| : I \in X\}.$$

Proof. Recall that if $(A_\lambda)_{\lambda \in \Lambda}$ is a family of C*-algebras then the direct sum $\oplus_{\lambda \in \Lambda} A_\lambda$ is defined to be the set of all $(a_\lambda)_{\lambda \in \Lambda}$ in $\Pi_{\lambda \in \Lambda} A_\lambda$ such that $\sup_{\lambda \in \Lambda} \|a_\lambda\|$ exists. This is a C*-algebra under pointwise defined operations and norm

$$\|(a_\lambda)\| = \sup_{\lambda \in \Lambda} \|a_\lambda\| \text{ for } a \in A.$$

Let $\theta : A/J \to \oplus_{I \in X} A/I$ be the natural map given by

$$\theta(a_J) = (a_I)_{I \in X}$$

for a in A. This is well defined because, if a is an element of J, then it is an element of each I in X, and because $\{\|a_I\| : I \in X\}$ is bounded above by $\|a_J\|$, so that $\sup_{I \in X} \|a_I\|$ exists.

It is easy to check that θ is a *-homomorphism. If $\theta(a_J)$ equals $\theta(b_J)$ then $a - b$ is an element of each I in X so a_J equals b_J and θ is injective. Thus θ is a *-isomorphism onto its image and hence an isometry. Therefore

$$\|a_J\| = \|\theta(a_J)\| = \sup\{\|a_I\| : I \in X\}.$$

□

The following may be found in [25], Theorem 4.9.14,

Theorem 2.16. *Let A be a C*-algebra, let a be an element of A and let C be a closed subset of* $\operatorname{Prim} A$. *Then there exists Q in C such that*

$$\|a_Q\| = \sup\{\|a_P\| : P \in C\}.$$

The following is an immediate corollary of Lemma 2.15 and Theorem 2.16.

Corollary 2.17. *Let A be a C*-algebra, let P be a primitive ideal of A and let G be a Glimm ideal of A such that $\phi(P)$ is G. Then there exists Q in $[P]$ such that*

$$\|a_G\| = \sup\{\|a_R\| : R \in [P]\} = \|a_Q\|.$$

The following theorem may be found in [24], paragraph 4.4.4.

Theorem 2.18. *Let A be a unital C*-algebra. Then for a in A and λ a strictly positive real number, the sets $\{P \in \operatorname{Prim} A : \|a_P\| \geq \lambda\}$ are compact.*

Let A be a unital C*-algebra, let a be an element of A and let λ be a strictly positive real number. If P is a primitive ideal and $\|a_P\| \geq \lambda$ then $\phi(P)$ is a subset of P and

$$\left\|a_{\phi(P)}\right\| \geq \|a_P\| \geq \lambda.$$

Conversely, if G is a Glimm ideal such that $\|a_G\| \geq \lambda$, we can take P in $\phi^{-1}(\{G\})$ such that

$$\|a_P\| = \|a_G\| \geq \lambda.$$

Therefore

$$\phi(\{P \in \operatorname{Prim} A : \|a_P\| \geq \lambda\}) = \{G \in \operatorname{Glimm} A : \|a_G\| \geq \lambda\}.$$

Since ϕ is continuous, it follows that this set is compact.

Recall that a real-valued function f on a topological space is said to be upper semi-continuous if $f^{-1}([\lambda, \infty))$ is closed for λ in $\mathbb{R}$. For each element a of A, define $\Phi_a : \operatorname{Glimm} A \to \mathbb{R}$ by

$$\Phi_a(G) = \|a_G\|$$

for G in $\operatorname{Glimm} A$. Then

$$\Phi_a^{-1}([\lambda, \infty)) = \{G \in \operatorname{Glimm} A : \|a_G\| \geq \lambda\}$$

and this set is closed since it is a compact subset of a Hausdorff space. Thus Φ_a is upper semi-continuous.

Chapter 3

The Distance to the Scalars

In this chapter we give an exposition of the main results of [33]. In particular we show that every element a of a unital C*-algebra A has a closest scalar $\lambda(a)$ and we calculate $\lambda(a)$ and the distance from a to $\lambda(a)$ for some examples. We also find expressions for $d(a, Z(A))$ and $\|\mathrm{ad}_A\, a\|$.

The following corollary is an exercise from [31].

Corollary 3.1. *Let A be a C*-algebra and let X be a non-empty subset of* $\mathrm{Id}\, A$ *such that the kernel of X is zero. Then, for all a in A,*

$$\|\mathrm{ad}_A\, a\| = \sup\left\{\left\|\mathrm{ad}_{A/I}\, a_I\right\| : I \in X\right\}.$$

Proof. Let J be an element of $\mathrm{Id}\, A$ and let b_J be an element of A/J_1. Then for all j in J,

$$\|[a_J, b_J]\| = \|[a, b+j]_J\| \leq \|[a, b+j]\| \leq \|\mathrm{ad}_A\, a\| \, \|b+j\|.$$

Thus,

$$\|[a_J, b_J]\| \leq \|\mathrm{ad}_A\, a\| \inf_{j \in J} \|b+j\| \leq \|\mathrm{ad}_A\, a\|,$$

which implies that $\left\|\mathrm{ad}_{A/J}\, a_J\right\|$ is bounded above by $\|\mathrm{ad}_A\, a\|$. It follows that

$$\sup\left\{\left\|\mathrm{ad}_{A/I}\, a_I\right\| : I \in X\right\} \leq \|\mathrm{ad}_A\, a\|.$$

Conversely, if b lies in the unit ball of A, then

$$\|[a, b]_I\| \leq \left\|\mathrm{ad}_{A/I}\, a_I\right\|$$

for all I in X. Applying Lemma 2.15,

$$\|[a, b]\| \leq \sup\left\{\left\|\mathrm{ad}_{A/I}\, a_I\right\| : I \in X\right\}.$$

Therefore,

$$\|ad_A a\| \leq \sup\left\{\left\|\mathrm{ad}_{A/I}\, a_I\right\| : I \in X\right\}.$$

□

In [33], Stampfli defined the **maximal numerical range** of a bounded linear operator T on a Hilbert space H to be

$$W_0(T) = \{\lambda \in \mathbb{C} : \exists\, (\xi_n) \subseteq H,\ \|\xi_n\| = 1,\ \langle T\xi_n, \xi_n\rangle \to \lambda,\ \|T\xi_n\| \to \|T\|\}.$$

In [14], Fong defined the **(algebraic) maximal numerical range** of an element a of a unital C*-algebra A to be

$$V_A^0(a) = \left\{x(a) : x \in S_A^0(a)\right\},$$

where

$$S_A^0(a) = \left\{x \in S_A : x(a^*a) = \|a\|^2\right\}$$

is the set of **maximal states** of a. The **(algebraic) numeric range** is defined by

$$V_A(a) = \{x(a) : x \in S_A\}.$$

Clearly $V_A^0(a)$ is a subset of $V_A(a)$. It was shown in [14] that

$$W_0(T) = V_{B(H)}^0(T).$$

In [33], the maximal numerical range was used to prove the **Pythagorean relation** for operators; that if T is an element of $B(H)$ then there exists a unique complex number λ such that for all complex numbers μ,

$$\|T-\lambda\|^2 + |\lambda-\mu|^2 \le \|T-\mu\|^2. \tag{3.1}$$

The equation

$$\left\|\mathrm{ad}_{B(H)} T\right\| = 2\|T-\lambda\| \tag{3.2}$$

was then proved.

By the Gelfand-Naimark Theorem, 3.1 holds for any element of a C*-algebra. The same cannot be deduced for 3.2, since the Gelfand-Naimark Theorem only gives that A is isometrically *-isomorphic to a C*-subalgebra of $B(H)$ for some Hilbert space H. Hence $\|\mathrm{ad}_A a\|$ may be less than $\left\|\mathrm{ad}_{B(H)} a\right\|$. However, as observed in [33], it does hold if A is primitive.

Stampfli's proof of the Pythagorean relation depends on the fact that $W_0(T)$ is convex, a non-trivial consequence of the Toeplitz-Hausdorff Theorem. We avoid this difficulty by proving the result for a general C*-algebra, using the definition of Fong. The idea of the proof is the same as that of Stampfli, but some of the details differ. This approach does not appear to have been used elsewhere. Unfortunately, this strategy does not appear to help with the second equation, and we need the equivalence of the definitions from [14] to complete the proof.

Lemma 3.2. *Let a be an element of a unital C*-algebra A. Then the spectrum of a, $\sigma_A(a)$, is a subset of $V_A(a)$, the numerical range, which in turn is a subset of the disc centred at the origin of radius $\|a\|$.*

Proof. Let λ be an element of $\sigma_A(a)$. Then $\lambda - a$ is not invertible. Let J be the left ideal $A(\lambda - a)$. Since J does not contain the unit, it is proper. Let b be an element of J. If $1-b$ lay in the unit ball, $1-(1-b) = b$ would be invertible, which contradicts the fact that J does not contain the unit. Therefore, $1-b$ does not lie in the unit ball for all b in J. In particular,

$$\|\mu + b\| \ge |\mu|$$

for all b in J and μ in $\mathbb{C}$. Define $f : J \oplus \mathbb{C}1 \to \mathbb{C}$ by

$$f(b+\mu) = \mu$$

for b in J and μ in $\mathbb{C}$. Then

$$\frac{|f(b+\mu)|}{\|b+\mu\|} \le 1$$

for all b in J and μ in $\mathbb{C}$ and $f(1)$ is 1. Hence f has unit norm. By the Hahn-Banach Theorem, f has an extension to a state x of A which is zero on J. In particular,

$$x(\lambda - a) = \lambda - x(a) = 0$$

so λ lies in $V_A(a)$. Since $|x(a)|$ is bounded above by $\|a\|$,

$$\sigma_A(a) \subseteq V_A(a) \subseteq B_{\|a\|}(0).$$

□

Since a^*a is self adjoint, $r_A(a^*a)$ equals $\|a\|^2$, and because a^*a it is positive $\sigma_A(a^*a)$ lies in $\mathbb{R}^+$. Since $\sigma_A(a^*a)$ is closed, $\|a\|^2$ lies in $\sigma_A(a^*a)$. Hence, a corollary of Lemma 3.2 is that there exists x in S_A such that $x(a^*a)$ equals $\|a\|^2$. Thus $S_A^0(a)$ and $V_A^0(a)$ are non-empty and

$$\|a\| = \sup\left\{x^{\frac{1}{2}}(a^*\,a) : x \in S_A\right\}.$$

Since $S_A^0(a)$ is the inverse image of the singleton $\|a\|^2$ under the map $\widehat{a^*a}$ restricted to S_A, $S_A^0(a)$ is weak*-closed and therefore weak*-compact. Since $V_A^0(a)$ is the image of $S_A^0(a)$ under $\hat{a}$, it is also compact. It is immediate that $S_A^0(a)$, and hence $V_A^0(a)$, are convex. In fact $S_A^0(a)$ is easily seen to be a face of S_A, although we shall not use this here. We may now prove a lemma equivalent to [33], Theorem 2.

Lemma 3.3. *Let a be an element of a unital C*-algebra A. If $V_A^0(a)$ contains the origin then*

$$\|a\|^2 + |\mu|^2 \leq \|a+\mu\|^2$$

for all complex numbers μ. Conversely, if

$$\|a\| \leq \|a+\mu\|$$

for all complex numbers μ, then $V_A^0(a)$ contains the origin.

Proof. If $V_A^0(a)$ contains the origin then there exists x in $S_A^0(a)$ such that $x(a)$ is zero. Recall that x is self-adjoint, and hence that $x(a^*)$ is zero. Then for all μ in $\mathbb{C}$

$$\begin{aligned}\|a+\mu\|^2 &\geq x\left((a+\mu)^*(a+\mu)\right)\\ &= x(a^*a) + \overline{\mu}x(a) + \mu x(a^*) + |\mu|^2\\ &= \|a\|^2 + |\mu|^2\ .\end{aligned}$$

This proves the first statement.

Suppose that a is such that $\|a\| \leq \|a+\mu\|$ for all complex numbers μ. Assume that $V_A^0(a)$ does not contain the origin. Let θ be an element of $[0, 2\pi]$. By linearity of states, $V_A^0\left(e^{i\theta}a\right)$ is the image of $V_A^0(a)$ under an anticlockwise rotation about the origin by θ. Since $V_A^0(a)$ is closed and convex there exists a θ in $[0, 2\pi]$ such that, for all λ in $V_A^0\left(e^{i\theta}a\right)$, $\operatorname{Re}\lambda \geq \tau$ for some $\tau > 0$. Let $b = e^{i\theta}a$.

Define G to be the set $\left\{x \in S_A : \operatorname{Re} x(b) \leq \frac{\tau}{2}\right\}$. Since G is the inverse image of $\left[-\|b\|, \frac{\tau}{2}\right]$ under $\operatorname{Re}\circ\widehat{b}$ restricted to S_A, G is a weak*-closed subset of S_A and is hence weak*-compact. If G is non-empty, the set

$$\left\{x^{\frac{1}{2}}(b^*b) : x \in G\right\}$$

is bounded above by $\|b\|$ and hence has supremum η. Since G is weak*-compact, η is achieved on G. Assume that $\eta = \|b\|$. Then there exists x in G such that $\|b\|^2 = x(b^*b)$. But then,

$$\operatorname{Re} x(b) \leq \frac{\tau}{2} < \tau,$$

which contradicts the fact that $x(b)$ lies in $V_A(b)$. Therefore $0 \leq \eta < \|b\|$. If G is empty, let η be zero.

Define a strictly positive number ν by

$$\nu = \min\left\{\frac{\tau}{2}, \frac{\|b\|-\eta}{2}\right\}.$$

If x is a state in the complement of G, then $\operatorname{Re} x(b) > \frac{\tau}{2}$ which implies $-2\nu\operatorname{Re} x(b) < -\nu\tau$. Thus

$$\begin{aligned}x\left((b-\nu)^*(b-\nu)\right) &= x(b^*b) - 2\nu\operatorname{Re} x(b) + \nu^2\\ &< x(b^*b) - \nu\tau + \nu^2\\ &= x(b^*b) + \nu(\nu-\tau)\,.\end{aligned}$$

Since $0 < \nu \leq \frac{\tau}{2}$, it follows that

$$\nu(\nu - \tau) \leq -\frac{\tau\nu}{2}.$$

Therefore,

$$x\left((b-\nu)^*(b-\nu)\right) < x(b^*b) - \frac{\tau\nu}{2} \leq \|b\|^2 - \varepsilon_1$$

where ε_1 is the strictly positive number $\tau\nu/2$. If G is non-empty, let x be an element of G. Then

$$-2\nu \operatorname{Re} x(b) \leq 2\nu |x(b)| \leq 2\nu x^{\frac{1}{2}}(b^*b) \leq 2\eta\nu.$$

Therefore,

$$x\left((b-\nu)^*(b-\nu)\right) < \eta^2 + 2\eta\nu + \nu^2 = (\eta+\nu)^2 \leq \left(\frac{\|b\|+\eta}{2}\right)^2 = \|b\|^2 - \varepsilon_2$$

where ε_2 is the strictly positive number $\frac{1}{4}(\|b\| - \eta)(3\|b\| + \eta)$. If G is empty put $\varepsilon_2 = \varepsilon_1$. Then we have shown that, for all x in S_A,

$$x\left((b-\nu)^*(b-\nu)\right) \leq \|b\|^2 - \min\{\varepsilon_1, \varepsilon_2\},$$

from which it follows that $\|b - \nu\| < \|b\|$. But then $\left\|a - e^{i\theta}\nu\right\| < \|a\|$ which contradicts the hypothesis. Therefore $V_A^0(a)$ contains the origin. □

Let a and b be elements of A such that $a - b$ is scalar. Then

$$\|a-b\|^2 = x\left((a-b)^*(a-b)\right)$$

for all x in S_A. Therefore,

$$\begin{aligned}
\|a+b\|^2 + \|a-b\|^2 &= \sup_{x\in S_A}\left\{x\left((a+b)^*(a+b) + x\left((a-b)^*(a-b)\right)\right)\right\} \\
&= \sup_{x\in S_A}\left\{2x(a^*a) + 2x(b^*b)\right\} \\
&\leq 2\sup_{x\in S_A}\left\{x(a^*a)\right\} + 2\sup_{x\in S_A}\left\{x(b^*b)\right\} \\
&= 2\left(\|a\|^2 + \|b\|^2\right).
\end{aligned}$$

Now let (λ_n) be a sequence in $\mathbb{C}$ such that $\|a - \lambda_n\|$ converges to $d(a, \mathbb{C}1)$. Let N_ε be such that

$$\|a - \lambda_n\| < d(a, \mathbb{C}1) + \varepsilon$$

for all $n \geq N_\varepsilon$. Then for $n, m \geq N_\varepsilon$, by the inequality above,

$$\|2a - \lambda_n - \lambda_m\|^2 + |\lambda_n - \lambda_m|^2 \leq 2\|a - \lambda_n\|^2 + 2\|a - \lambda_m\|^2.$$

Therefore,

$$|\lambda_n - \lambda_m|^2 \leq 4(d(a, \mathbb{C}1) + \varepsilon)^2 - 4d(a, \mathbb{C}1)^2 = 4\varepsilon(2d(a, \mathbb{C}1) + \varepsilon).$$

Thus (λ_n) is Cauchy and converges to some complex number $\lambda(a)$. By norm-continuity

$$\|a - \lambda(a)\| = \lim_{n\to\infty}\|a - \lambda_n\| = d(a, \mathbb{C}1).$$

Theorem 3.4. *Let A be a unital C^*-algebra and let a be an element of A. Then there exists a unique complex number $\lambda(a)$ such that, for all complex numbers μ,*

$$\|a - \lambda(a)\|^2 + |\lambda(a) - \mu|^2 \leq \|a - \mu\|^2.$$

Proof. Let $\lambda(a)$ be a complex number such that

$$\|a - \lambda(a)\| = d(a, \mathbb{C}1).$$

Then, for all μ in $\mathbb{C}$,

$$\|a - \lambda(a)\| \leq \|(a - \lambda(a)) + \mu\|.$$

Then, by Lemma 3.3, the origin lies in $V_A^0(a - \lambda(a))$. Hence, for all μ in $\mathbb{C}$,

$$\|a - \lambda(a)\|^2 + |\mu|^2 \leq \|(a - \lambda(a)) + \mu\|^2 \ .$$

Thus, for all μ in $\mathbb{C}$,

$$\|a - \lambda(a)\|^2 + |\lambda(a) - \mu|^2 \leq \|a - \mu\|^2.$$

Suppose that λ' is another complex number satisfying the inequality. Then

$$\|a - \lambda(a)\|^2 + 2|\lambda(a) - \lambda'|^2 \ \leq \|a - \lambda'\|^2 + |\lambda(a) - \lambda'|^2 \leq \|a - \lambda(a)\|^2.$$

Therefore $|\lambda(a) - \lambda'|$ is zero, which implies that $\lambda(a)$ is unique. $\square$

If X is a compact subset of $\mathbb{C}$ then there exists a unique circle of minimum radius containing X. This circle is called the **circumcircle** of X, its centre is the **circumcentre** of X and its radius the **circumradius** of X. If S is the circumcircle of X, then the following two fundamental properties of circumcircles are satisfied:

(i) every closed semi-circle of S intersects X;

(ii) there exist x, y and z in $X \cap S$ (not necessarily distinct) such that S is the circumcircle of $\{x, y, z\}$.

Lemma 3.5. *Let X be a compact subset of C, and let r be the circumradius of the circumcircle S of X. Then there exist x and y in $X \cap S$ such that*

$$|x - y| \geq \sqrt{3}r.$$

Proof. Let μ be the circumcentre of X. We first show that there exist x and y in $X \cap S$ such that $\angle x\mu y$, the angle between x and y through μ, lies between $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ radians. Let u be an element of $S \cap X$. If there exists v in $S \cap X$ such that $\angle u\mu v$ lies between $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ radians then there is nothing more to do. Otherwise, by property (ii), there exist w and z in $X \cap S$ in the regions indicated in Figure 3.1. Clearly $\angle w\mu z$ lies between $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ radians so the statement is proven.

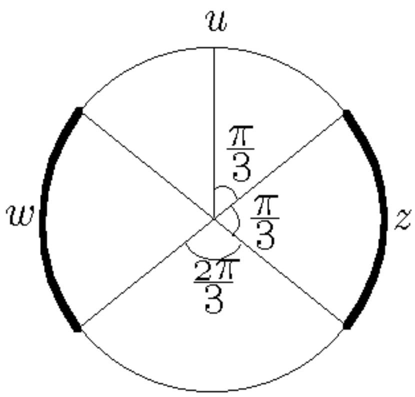

Figure 3.1: The positions of w and z.

Let θ be the angle between two such elements, x and y. Then $\cos\theta \leq \frac{1}{2}$ so, by the cosine rule,

$$\begin{aligned}|x-y|^2 &= 2r - 2r^2\cos\theta \\ &= 2r\left(1-\cos\theta\right) \\ &\geq 3r^2.\end{aligned}$$

This completes the proof. □

For a in A, the spectrum of a is a compact subset of $\mathbb{C}$, and hence has a unique circumcircle, centre C_a and radius R_a. Clearly $r_A(a) \geq R_a$. Let a be normal. Then $a-\mu$ is normal and

$$r_A(a-\mu) = \|a-\mu\|, \qquad \sigma_A(a-\mu) = \sigma_A(a) - \mu.$$

Thus,

$$R_a = R_{a-\mu}, \qquad C_{a-\mu} = C_a - \mu,$$

for μ in $\mathbb{C}$. Therefore C_{a-C_a} is zero, which implies

$$\|a-C_a\| = r_A(a-C_a) = R_a = R_{a-\mu} \leq r_A(a-\mu) = \|a-\mu\|.$$

Hence,

$$R_a = \|a-C_a\| = d(a,\mathbb{C}1) = \|a-\lambda(a)\|.$$

But then, by Theorem 3.4,

$$\|a-\lambda(a)\|^2 + |\lambda(a)-C_a|^2 \leq \|a-C_a\|^2 = \|a-\lambda(a)\|^2,$$

and $\lambda(a)$ equals C_a. Thus we have proved that, for a normal, $\lambda(a)$ is the circumcentre of $\sigma_A(a)$, whilst $d(a,\mathbb{C}1)$ is the circumradius of $\sigma_A(a)$. Furthermore, if a is self-adjoint, then its spectrum is real, non-empty and bounded and, hence,

$$\alpha = \sup\sigma_A(a), \qquad \beta = \inf\sigma_A(a),$$

exist. Then

$$\begin{aligned}\lambda(a) = C_a &= \frac{1}{2}(\alpha+\beta) \\ \|a-\lambda(a)\| = d(a,\mathbb{C}1) = R_a &= \frac{1}{2}(\alpha-\beta).\end{aligned}$$

We now use these facts to calculate $\lambda(a)$ and $d(a,\mathbb{C}1)$ in some particular cases. These results will be used in later proofs.

For an element a in A let $\imath : \sigma_A(a) \to \mathbb{C}$ be the inclusion map. Let a be an element of A_{sa} so that $\sigma_A(a)$ is a subset of $\mathbb{R}$ and define

$$\imath_+ = \max\{0,\imath\}, \qquad \imath_- = \max\{0,-\imath\}$$

These real-valued functions are continuous and

$$\imath = \imath_+ - \imath_-, \qquad \imath_+\imath_- = 0$$

so by the functional calculus there exist elements a_+ and a_- in A_{sa} such that

$$a = a_+ - a_-, \qquad a_+a_- = 0.$$

By the spectral mapping theorem

$$\sigma_A(a_+) = \imath_+(\sigma_A(a)), \qquad \sigma_A(a_-) = \imath_-(\sigma_A(a)).$$

Thus $\sigma_A(a_+)$ and $\sigma_A(a_-)$ are subsets of $\mathbb{R}^+$ so a_+ and a_- are positive. This is the well known orthogonal decomposition of a in A_{sa} [27], Definition 1.4.3. Now $|\imath| = \imath_+ + \imath_-$ so we define $|a| = a_+ + a_-$. Since

$$\begin{aligned} |\imath|^2 &= (\imath_+ + \imath_-)^2 \\ &= \imath_+^2 + \imath_-^2 \\ &= (\imath_+ - \imath_-)^2 \\ &= \imath^2 \end{aligned}$$

it follows from uniqueness of positive square roots that

$$|a| = (a^*a)^{\frac{1}{2}} = \left(a^2\right)^{\frac{1}{2}}.$$

We may write

$$\imath_+ = \frac{1}{2}(|\imath| + \imath), \qquad \imath_- = \frac{1}{2}(|\imath| - \imath)$$

from which it follows that

$$a_+ = \frac{1}{2}(|a| + a), \qquad a_- = \frac{1}{2}(|a| - a).$$

If $a = b - c$ for some b and c in A^+ such that bc is zero then

$$a^*a = (b+c)^2$$

so taking unique positive square roots $|a| = b + c$. Thus $a_+ = b$ and $a_- = c$ so the orthogonal decomposition is unique.

Suppose that a in A_{sa} is such that a_+ and a_- both have unit norm. Since a_+ and a_- are positive, it follows that

$$\alpha(a_+) = \alpha(a_-) = 1,$$

and hence that

$$\alpha(a) = 1, \qquad \beta(a) = -1.$$

Thus,

$$\lambda(a) = 0, \qquad d(a, \mathbb{C}1) = 1.$$

Now let a be a normal element of A, let $\alpha = e^{\frac{2\pi i}{3}}$ and define functions $\imath_0$, $\imath_1$ and $\imath_2$ on $\mathbb{C}$ by

$$\begin{aligned} \imath_0 &= \frac{1}{3}(|\imath| + \imath + \imath^*) \\ \imath_1 &= \frac{1}{3}(|\imath| - \imath + \alpha(\imath^* - \imath)) \\ \imath_2 &= \frac{1}{3}(|\imath| - \imath + \bar{\alpha}(\imath^* - \imath)). \end{aligned}$$

If $\lambda = x + iy$ is a complex number, with x and y the real and imaginary parts, we have

$$\begin{aligned} \imath_0(\lambda) &= \frac{1}{3}(|\lambda| + 2x) \\ \imath_1(\lambda) &= \frac{1}{3}\left(|\lambda| - x + \sqrt{3}y\right) \\ \imath_2(\lambda) &= \frac{1}{3}\left(|\lambda| - x - \sqrt{3}y\right) \end{aligned}$$

so $\imath_0$, $\imath_1$ and $\imath_2$ are real-valued. Direct calculation shows that

$$\imath = \imath_0 + \alpha\imath_1 + \alpha^2\imath_2.$$

By restricting to $\sigma_A(a)$ and applying the functional calculus, there exist elements a_0, a_1 and a_2 in A_{sa} such that

$$a = a_0 + \alpha a_1 + \alpha^2 a_2.$$

Let us suppose that a is such that a_0, a_1 and a_2 are positive and have zero pairwise products. Then arguing as in the orthogonal decomposition shows that this decomposition of a is unique. An elementary calculation shows that

$$\imath_0(\lambda) = 0 \text{ if and only if } x \leq 0 \text{ and } y = \pm\sqrt{3}x$$
$$\imath_1(\lambda) = 0 \text{ if and only if } x \leq 0 \text{ and } y = \sqrt{3}x \text{ or } x \geq 0 \text{ and } y = 0$$
$$\imath_2(\lambda) = 0 \text{ if and only if } x \leq 0 \text{ and } y = -\sqrt{3}x \text{ or } x \geq 0 \text{ and } y = 0$$

By continuity of $\imath_0$, the kernel of $\imath_0$ divides the complex plane into two regions, with $\imath_0$ strictly positive on one and strictly negative on the other. Similarly for $\imath_1$ and $\imath_2$. Thus $\sigma_A(a)$ must be a subset of the intersection of the non-negative regions for $\imath_0$, $\imath_1$ and $\imath_2$. Hence $\sigma_A(a)$ is a subset of the union of the lines L_0, L_1 and L_2 in the complex plane, where L_0 is the non-negative real axis, L_1 is the half line from the origin through α and L_2 is the half line from the origin through $\bar{\alpha}$ as shown in Figure 3.2. Furthermore ι_j restricted to $\sigma_A(a)$ is zero except on $\sigma_A(a) \cap L_j$.

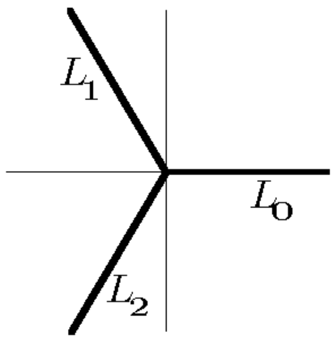

Figure 3.2: The sets L_0, L_1 and L_2.

Let λ be a non-negative real number. Then

$$\iota_0(\lambda 1) = \lambda, \qquad \iota_1(\lambda\alpha) = \lambda, \qquad \iota_2(\lambda\bar{\alpha}) = \lambda$$

so each ι_j rotates $\sigma_A(a) \cap L_j$ about the origin to the non-negative real line and maps the rest of $\sigma_A(a)$ to the origin. Since $\|a_j\|$ is positive it follows that $\|a_j\|\alpha^j$ lies in $\sigma_A(a)$ and $\sigma_A(a) \cap L_j$ is a subset of $\left[0, \|a_j\|\alpha^j\right]$. Thus, the circumcircle of $\sigma_A(a)$ is the circumcircle of $\{\|a_0\| 1, \|a_1\|\alpha, \|a_2\|\bar{\alpha}\}$. In particular, if the a_j are of unit norm, $\lambda(a)$ is zero and $d(a, \mathbb{C}1)$ is 1. If a_0 is zero but a_1 and a_2 lie in the unit ball, then the circle with centre $-\frac{1}{2}$ and radius $\sqrt{3}/2$ contains 0, α and $\bar{\alpha}$, and $d(a, \mathbb{C}1)$ is bounded above by $\sqrt{3}/2$.

Let A be a unital C*-algebra, let I be an ideal of A and let a be a self adjoint element of A. By the spectral mapping theorem, $\|a\| + a_I$ and $\|a\| - a_I$ are positive elements of A, and therefore

$$\alpha(\|a\| + a_I) = \|\|a\| + a_I\|, \qquad \alpha(\|a\| - a_I) = \|\|a\| - a_I\|.$$

Clearly

$$\alpha(\|a\| + a_I) = \|a\| + \alpha(a_I), \qquad \alpha(\|a\| - a_I) = \|a\| - \beta(a_I),$$

and therefore,

$$\alpha(a_I) = \|\|a\| + a_I\| - \|a\|, \qquad \beta(a_I) = \|a\| - \|\|a\| - a_I\|.$$

Let G be a Glimm ideal of A. Then, by Corollary 2.17, there exist primitive ideals P and Q of A such that $\phi(P)$ and $\phi(Q)$ are G and

$$\|\|a\| + a_P\| = \|\|a\| + a_G\|, \qquad \|\|a\| + a_Q\| = \|\|a\| + a_G\|.$$

Thus,

$$\begin{aligned}\alpha(a_G) &= \| \|a\| + a_G\| - \|a\| \\ &= \| \|a\| + a_P\| - \|a\| \\ &= \alpha(a_P).\end{aligned}$$

Similarly, $\beta(a_G) = \beta(a_Q)$. Let R be the intersection of P and Q. Then

$$\begin{aligned}\alpha(a_R) &= \| \|a\| + a_R\| - \|a\| \\ &= \max\{\| \|a\| + a_P\|, \| \|a\| + a_Q\|\} - \|a\| \\ &= \| \|a\| + a_G\| - \|a\| \\ &= \alpha(a_G).\end{aligned}$$

Similarly $\beta(a_R) = \beta(a_G)$.

The following theorem is known as the Kadison Transitivity Theorem. Its proof may be found in [16], Theorem 1.

Theorem 3.6. *Let A be a primitive C^*-algebra with faithful irreducible representation (π, H). Let V be an element of $\mathcal{U}(B(H))$ and let $\{\xi_1, \ldots, \xi_n\}$ be a subset of H. Then there exists u in $\mathcal{U}(A)$ such that*

$$\pi(u)\xi_k = V\xi_k \qquad k = 1 \ldots n$$

and

$$\sigma_A(u) \neq \{\lambda \in \mathbb{C} : |\lambda| = 1\}.$$

The following is proved in [33], Lemma 3 and Theorem 5.

Lemma 3.7. *Let A be a primitive C^*-algebra, let a be an element of A and let λ be an element of $V_A^0(a)$. Then*

$$\|\mathrm{ad}_A\, a\| \geq 2\left(\|a\|^2 - |\lambda|^2\right)^{\frac{1}{2}}.$$

Proof. Since it is primitive, A has a faithful irreducible representation (π, H). Identifying A with its image and applying the Hahn-Banach theorem gives

$$V_A^0(a) = V_{B(H)}^0(a) = W_0(a).$$

Thus there exists a sequence (ξ_n) in H such that each ξ_n has unit norm, $(\|a\xi_n\|)$ converges to $\|a\|$ and $(\langle a\xi_n, \xi_n\rangle)$ converges to λ. Write

$$a\xi_n = \alpha_n\xi_n + \beta_n\eta_n,$$

where

$$\alpha_n = \langle a\xi_n, \xi_n\rangle, \qquad \beta_n = \|(a - \alpha_n)\xi_n\|, \qquad \eta_n = \frac{1}{\beta_n}(a - \alpha_n)\xi_n.$$

Then η_n has unit norm, $\langle \xi_n, \eta_n\rangle$ is zero and ξ_n, η_n are linearly independent. Since

$$H = \langle \xi_n\rangle \oplus \langle \eta_n\rangle \oplus \{\xi_n, \eta_n\}^{\perp},$$

we may define $V_n : H \to H$ by

$$V_n(\alpha\xi_n + \beta\eta_n + \zeta) = \alpha\xi_n - \beta\eta_n + \zeta,$$

where ζ lies in $\{\xi_n, \eta_n\}^{\perp}$. Clearly V_n is linear and

$$\begin{aligned}\|V_n(\alpha\xi_n + \beta\eta_n + \zeta)\|^2 &= \|\alpha\xi_n - \beta\eta_n + \zeta\|^2 \\ &= |\alpha|^2 + |\beta|^2 + \|\zeta\|^2 \\ &= \|\alpha\xi_n + \beta\eta_n + \zeta\|^2.\end{aligned}$$

Therefore V_n is a surjective isometry and hence unitary. Applying Theorem 3.6, there exists u_n in $\mathcal{U}(A)$ such that

$$u_n \xi_n = \xi_n,\ u_n \eta_n = -\eta_n.$$

Thus,

$$u_n a \xi_n = 2\alpha_n \xi_n - a\xi_n,$$

and

$$\|\mathrm{ad}_A\, a\| \geq \|(au_n - u_n a)\, \xi_n\| = 2\, \|(a - \alpha_n)\, \xi_n\|\,.$$

Furthermore,

$$\begin{aligned}\|(a-\alpha_n)\,\xi_n\|^2 &= \|a\xi_n\|^2 - \overline{\alpha_n}\,\langle \alpha_n \xi_n, \xi_n\rangle - \alpha_n\,\langle \xi_n, \alpha_n \xi_n\rangle + |\alpha_n|^2 \\ &= \|a\xi_n\|^2 - |\alpha_n|^2\end{aligned}$$

so

$$\|\mathrm{ad}_A\, a\| \geq 2\left(\|a\xi_n\|^2 - |\alpha_n|^2\right)^{\frac{1}{2}}.$$

Letting n tend to infinity, $(\|a\xi_n\|)$ converges to $\|a\|$ and (α_n) converges to λ. Hence

$$\|\mathrm{ad}_A\, a\| \geq 2\left(\|a\|^2 - |\lambda|^2\right)^{\frac{1}{2}}.$$

□

Theorem 3.8. *Let A be a primitive unital C^*-algebra. Then, for all a in A,*

$$\|\mathrm{ad}_A\, a\| = 2\, \|a - \lambda(a)\|\,.$$

Proof. For all μ in $\mathbb{C}$

$$\|a - \lambda(a)\| \leq \|(a - \lambda(a)) + \mu\|$$

and, by Lemma 3.3, $V_A^0(a - \lambda(a))$ contains the origin. Therefore, by Lemma 3.7,

$$\|\mathrm{ad}_A\, a\| = \|\mathrm{ad}_A\,(a - \lambda(a))\| \geq 2\, \|a - \lambda(a)\|\,.$$

Conversely,

$$\|\mathrm{ad}_A\, a\| = \|\mathrm{ad}_A\,(a - \lambda(a))\| \leq 2\, \|a - \lambda(a)\|\,.$$

□

The following is an immediate corollary of Theorem 3.8 and Corollary 3.1.

Corollary 3.9. *Let A be a unital C^*-algebra. Then, for all a in A,*

$$\|\mathrm{ad}_A\, a\| = 2 \sup\{\|a_P - \lambda(a_P)\| : P \in \mathrm{Prim}\, A\}\,.$$

Proof. By Theorem 3.8 and Corollary 3.1,

$$\begin{aligned}\|\mathrm{ad}_A\, a\| &= \sup\left\{\left\|\mathrm{ad}_{A/P}\, a_P\right\| : P \in \mathrm{Prim}\, A\right\} \\ &= 2\sup\{\|a_P - \lambda(a_P)\| : P \in \mathrm{Prim}\, A\}\,.\end{aligned}$$

□

For the next proof we use the following theorem, which can be found in [26], Theorem 2.13.

Theorem 3.10. *Let $U_1, \ldots, U_n$ be open subsets of a locally compact Hausdorff space X and let M be a compact subset of $U_1 \cup \ldots \cup U_n$. Then there exists a partition of unity on M subordinate to $U_1, \ldots, U_n$, that is to say continuous functions $f_1, \ldots, f_n$ on X with compact support such that, for x in X,*

$$0 \leq f_j(x) \leq 1, \qquad \mathrm{supp}\, f_j \subseteq V_j, \qquad j = 1, \ldots, n$$

and

$$f_1(x) + \ldots + f_n(x) = 1$$

for all x in M.

We may now prove [31], Theorem 2.3.

Theorem 3.11. *Let A be a unital C^*-algebra. Then, for a in A,*

$$d(a, Z(A)) = \sup\{\|a_G - \lambda(a_G)\| : G \in \operatorname{Glimm} A\}.$$

Proof. For G in $\operatorname{Glimm} A$ let $p_G : A \to A/G$ be the canonical map . Clearly $p_G(Z(A))$ contains the scalars. Conversely, let z_G be an element of $p_G(Z(A))$, where z lies in $Z(A)$, and recall that

$$Z(A) = (G \cap Z(A)) \oplus \mathbb{C}1_G.$$

Then z_G is a scalar and $p_G(Z(A))$ is $\mathbb{C}1_G$. It then follows from Theorem 3.4 that, for all z in $Z(A)$,

$$\|a - z\| \geq \|(a - z)_G\| \geq \|a_G - \lambda(a_G)\|.$$

Thus

$$\alpha = \sup\{\|a_G - \lambda(a_G)\| : G \in \operatorname{Glimm} A\}$$

exists and α is bounded above by $d(a, Z(A))$.

Recall from Chapter 2 that $\Phi_a : \operatorname{Glimm} A \to \mathbb{R}^+$, the map taking G to $\|a_G\|$, is upper semi-continuous, and, given $\varepsilon > 0$, the set

$$M = \Phi_a^{-1}([\alpha + \varepsilon, \infty))$$

is compact. If M is empty, then, by Lemma 2.15,

$$d(a, Z(A)) \leq \|a\| = \sup\{\|a_G\| : G \in \operatorname{Glimm} A\} \leq \alpha + \varepsilon.$$

Otherwise, for H in M define

$$U_H = \Phi_{a-\lambda(a_H)}^{-1}((-\infty, \alpha + \varepsilon)).$$

Since $\Phi_{a-\lambda(a_H)}$ is upper semi-continuous and

$$\Phi_{a-\lambda(a_H)}(H) = \|a_H - \lambda(a_H)\| \leq \alpha,$$

U_H is an open neighbourhood of H. Thus $\{U_H : H \in M\}$ is an open cover for M and, by compactness, there exist $H^1, \ldots, H^n$ in M such that $U_{H^1}, \ldots, U_{H^n}$ form an open sub-cover. Let $f^1, \ldots, f^n$ be a partition of unity for M, defined on $\operatorname{Glimm} A$ and subordinate to $U_{H^1}, \ldots U_{H^n}$. Let $z^1, \ldots, z^n$ be elements of $Z(A)$ such that, for all G in $\operatorname{Glimm} A$,

$$z_G^j = f^j(G) 1_G \qquad j = 1 \ldots n.$$

Let z be the element $\sum_{j=1}^n \lambda^j z^j$ of $Z(A)$ where λ^j denotes $\lambda(a_{H^j})$. Then, for each G in $\operatorname{Glimm} A$,

$$\begin{aligned} \|a_G - z_G\| &= \left\|a_G - \textstyle\sum_{j=1}^n \lambda^j z_G^j\right\| \\ &\leq \left\|a_G - \textstyle\sum_{j=1}^n z_G^j a_G\right\| + \left\|\textstyle\sum_{j=1}^n z_G^j a_G - \sum_{j=1}^n \lambda^j z_G^j\right\| \\ &\leq \left(1 - \textstyle\sum_{j=1}^n f^j(G)\right) \Phi_a(G) + \textstyle\sum_{j=1}^n f^j(G) \Phi_{a-\lambda^j}(G) \\ &< \left(1 - \textstyle\sum_{j=1}^n f^j(G)\right) \Phi_a(G) + \left(\textstyle\sum_{j=1}^n f^j(G)\right)(\alpha + \varepsilon) \end{aligned}$$

since f^j is zero unless G lies in U_{H^j}. If G lies in M then $\sum_{j=1}^n f^j(G)$ sums to 1. If G is not an element of M then $\Phi_a(G)$ is less than $\alpha + \varepsilon$. Therefore, in either case,

$$d(a, Z(A)) \leq \|a_G - z_G\| < \alpha + \varepsilon.$$

Since ε was arbitrary, the result follows. □

Chapter 4

Pure Functionals

We begin this chapter with a statement of the polar decomposition theorem for linear functionals, [10], Theorem 12.2.4 and Definition 12.2.8.

Theorem 4.1. *Let A be a unital C*-algebra, with dual A^* and second dual A^{**}. Then, for each x in A^*, there exists a unique pair $(u, |x|)$ with u an element of A and a partial isometry in A^{**}, and $|x|$ an element of A_+^* such that*

$$\| |x| \| = \|x\|\,,\ x(a) = |x|(ua)\,,\ \ |x|(a) = x(u^* a)\,,$$

for all a in A, and with uu^* equal to the support of $|x|$.

The functional $|x|$ is known as the absolute value of x.

For a unital C*-algebra A we define G_A, the set of **pure functionals** of A, to be $\partial_e A_1^*$, the extreme points of the unit ball of the dual of A. We now quote a characterisation of G_A from [6], Proposition 1.1.

Theorem 4.2. *Let A be a C*-algebra and let x be a linear functional. Then x is a pure functional if and only if there exists a non-zero irreducible representation (π, H) such that, for all a in A,*

$$x(a) = \langle \pi(a)\,\xi, \eta\rangle\,,$$

where ξ and η are unit vectors in H. Furthermore, when this is the case,

$$|x|(a) = \langle \pi(a)\,\xi, \xi\rangle$$

for all a in A, and $|x|$ is a pure state.

Let a be an element of a C*-algebra A. Then there exists an element x in S_A such that

$$\begin{aligned}\|a\|^2 &= x(a^* a)\\ &= \langle \pi_x(a)\,\xi_x, \pi_x(a)\,\xi_x\rangle\\ &= \|\pi_x(a)\,\xi_x\|^2\,.\end{aligned}$$

Define a function f on A by

$$f(b) = \langle \pi(b)\,\xi, \eta\rangle$$

for b in A, where

$$\eta = \frac{1}{\|a\|}\pi_x(a)\,\xi_x.$$

Then η is a unit vector and f is a pure functional on A. Thus, for every a in A, there exists an element f in G_A such that $f(a)$ equals $\|a\|$.

Let I be an ideal in A, let f be a pure functional on A/I, and let (π, H) be an irreducible representation of A/I such that, for all a_I in A/I,

$$f(a_I) = \langle \pi(a_I)\xi, \eta \rangle$$

for some unit vectors ξ and η in H. Then $\pi \circ p_I$ is a *-homomorphism from A to H and $\pi \circ p_I(A)$ equals $\pi(A)$, from which it follows that $\pi \circ p_I(A)'$ equals $\pi(A)'$. Thus $(\pi \circ p_I, H)$ is an irreducible representation of A and, for all a in A,

$$f \circ p_I(a) = \langle \pi \circ p_I(a)\xi, \eta \rangle.$$

Therefore, f in $G_{A/I}$ induces $f \circ p_I$ in G_A. Similarly, let f be a pure functional on A and let (π, H) be an irreducible representation of A such that, for all a in A,

$$f(a) = \langle \pi(a)\xi, \eta \rangle,$$

for some unit vectors ξ and η in H. Then if I is an ideal of A contained in $\ker \pi$, an irreducible *-homomorphism is induced on A/I by π and f induces a pure functional

$$f(a_I) = \langle \pi(a)\xi, \eta \rangle$$

on A/I.

Define a map $\Delta_A : G_A \to \partial_e S_A$ by

$$\Delta_A(x) = |x|$$

for x in G_A. Let G_A and $\partial_e S_A$ have the weak*-topologies induced by A^*. Recall that if a subset X of A^* has the weak*-topology and x is a point in X then every neighbourhood U of x contains an open set of the form

$$V_X(x; a_1, \ldots, a_n; \varepsilon) = \{y \in X : |x(a_j) - y(a_j)| < \varepsilon \text{ for } j = 1 \ldots n\}$$

for some n in $\mathbb{N}$, $a_1, \ldots, a_n$ in A and $\varepsilon > 0$.

Let U be an open set in G_A and let x be an element of $\Delta(U)$. Then there exists y in U such that $x = |y|$. Let $a_1, \ldots, a_n$ in A and $\varepsilon > 0$ be chosen so that $V = V_{G_A}(y; a_1, \ldots, a_n; \varepsilon)$ is a subset of U. By Theorem 4.2, there exists a non-zero irreducible representation (π, H) such that, for all a in A,

$$y(a) = \langle \pi(a)\xi, \eta \rangle, \qquad x(a) = \langle \pi(a)\xi, \xi \rangle$$

where ξ and η are unit vectors in H. By [24], Proposition 3.13.4, there exists a unitary u of A such that

$$\pi(u)\eta = \xi.$$

Then, for all a in A,

$$x(ua) = y(a).$$

Let $W = V_{\partial_e S_A}(x; ua_1, \ldots, ua_n; \varepsilon)$, let w be an element of W, and define $z : A \to \mathbb{C}$ by

$$z(a) = w(ua)$$

for a in A. Clearly z is a linear functional and, for all a in A,

$$\begin{aligned} z(a) &= \langle \pi_w(ua)\xi_w, \xi_w \rangle \\ &= \langle \pi_w(a)\xi_w, \pi_w(u^*)\xi_w \rangle. \end{aligned}$$

Since u is unitary, $\pi_w(u^*)$ is isometric, and hence $\pi_w(u^*)\xi_w$ has unit norm. It follows from Theorem 4.2 that z is a pure functional. Furthermore, since

$$|w(ua_j) - x(ua_j)| < \varepsilon, \qquad j = 1 \ldots n,$$

it follows that

$$|z(a_j) - y(a_j)| < \varepsilon, \qquad j = 1 \dots n.$$

Hence z is an element of V. Clearly $\Delta(z)$ equals w, and we have shown that

$$W \subseteq \Delta(V) \subseteq \Delta(U).$$

Since x in $\Delta(U)$ was arbitrary and W is an open neighbourhood of x, it follows that $\Delta(U)$ is open and hence Δ is open.

Let x be a pure functional, and let (π_1, H_1) and (π_2, H_2) be irreducible representations such that, for all a in A,

$$x(a) = \langle \pi_1(a)\xi_1, \eta_1 \rangle = \langle \pi_2(a)\xi_2, \eta_2 \rangle.$$

Then, for all a, b and c in A,

$$x(a^*bc) = \langle \pi_1(b)\pi_1(c)\xi_1, \pi_1(a)\eta_1 \rangle = \langle \pi_2(b)\pi_2(c)\xi_2, \pi_2(a)\eta_2 \rangle.$$

Let b be an element of $\ker \pi_1$. Then, by continuity,

$$\langle \pi_2(b)\pi_2(c)\xi_2, \eta \rangle = 0$$

for all η in H_2. Thus $\pi_2(b)$ is zero on $\pi_2(A)\xi_2$ and hence on H_2. By symmetry, π_1 and π_2 have the same kernel, and there is a well defined map $\Gamma_A : G_A \to \operatorname{Prim} A$ given by

$$\Gamma_A(x) = \ker \pi,$$

where (π, H) is an irreducible representation of A such that

$$x(a) = \langle \pi(a)\xi, \eta \rangle$$

for some unit vectors ξ and η in H. As an immediate consequence of Theorem 4.2, we see that, for all x in G_A,

$$\Gamma_A(|x|) = \Gamma_A(x)$$

and $\Gamma_A = \Gamma_A \circ \Delta_A$. Since, for every x in $\partial_e S_A$,

$$x(a) = \langle \pi_x(a)\xi_x, \xi_x \rangle$$

for all a in A, it is immediate that Γ_A restricted to $\partial_e S_A$ agrees with the natural map $\ker : \partial_e S_A \to \operatorname{Prim} A$, discussed in Chapter 2. As shown there, ker is open. Since Δ_A is also open, Γ_A is the composition of open maps and is therefore open.

In the following discussion a net $(x_\omega)_{\omega \in \Omega}$ is said to be a subnet of a net $(x_\lambda)_{\lambda \in \Lambda}$ if there exists a function $\varphi : \Omega \to \Lambda$ such that $x_{\varphi(\omega)} = x_\omega$ and for each λ in Λ there is a ω_0 in Ω such that $\varphi(\omega) \geq \lambda$ whenever $\omega \geq \omega_0$ [21], p70. Note that this definition differs from that given by some authors (e.g. [35]). The following technical result may be found in [13], Chapter II, 13.2.

Lemma 4.3. *Let X and Y be topological spaces and let $f : X \to Y$ be an open map. Then, whenever a net $(y_\lambda)_{\lambda \in \Lambda}$ converges to a point $f(x)$ in Y for some x in X there exists a subnet $(y_\omega)_{\omega \in \Omega}$ of $(y_\lambda)_{\lambda \in \Lambda}$ and a net $(x_\omega)_{\omega \in \Omega}$ of X such that, for all ω in Ω,*

$$f(x_\omega) = y_\omega$$

and (x_ω) converges to x.

Proof. Let N_x be the set of neighbourhoods of x directed by set inclusion. Let Ω be the product of Λ and N_x with the product direction. Let ω be the element (λ, U) in Ω. Since f is open, $f(U)$ is an open neighbourhood of $f(x)$, and there is therefore a λ_0 in Λ such that y_μ is in $f(U)$ for all $\mu \geq \lambda_0$. Choose λ_ω greater than or equal to λ and λ_0 and let φ be the map taking ω to λ_ω.

For λ in Λ, let $\omega_0 = (\lambda, X)$. Then $\varphi(\omega) \geq \lambda$ whenever $\omega \geq \omega_0$ Hence, defining y_ω to be y_{λ_ω} for ω in Ω gives a subnet of $(y_\lambda)_{\lambda \in \Lambda}$. For each ω in Ω, choose x_ω in U such that $f(x_\omega) = y_\omega$. If U is an open neighbourhood of x, let $\omega_0 = (\lambda, U)$ for some λ in Λ. Then x_ω lies in U when $\omega \geq \omega_0$ and (x_ω) is a net converging to x such that $f(x_\omega) = y_\omega$ for all ω in Ω. □

We now prove an important formula for $\|\mathrm{ad}_A\, a\|$ following the proof given in [30], Proposition 5.7. For an alternative proof see [31], Proposition 2.6.

Proposition 4.4. *Let A be a unital C*-algebra. Then, for all a in A,*

$$\|ad_A a\| = 2\sup\{\|a_P - \lambda(a_P)\| : P \in \mathrm{Primal}\, A\}.$$

Proof. Since every primitive ideal is prime, and therefore primal,

$$\sup\{\|a_P - \lambda(a_P)\| : P \in \mathrm{Primal}\, A\} \geq \sup\{\|a_P - \lambda(a_P)\| : P \in \mathrm{Prim}\, A\}.$$

Let I be a proper primal ideal. By Theorem 2.7 there exists a net $(P_\alpha)_{\alpha\in\mathcal{A}}$ in $\mathrm{Prim}\, A$ converging to every point in $\mathrm{hull}\, I$. By Theorem 3.4

$$|\lambda(a_{P_\alpha})| \leq \|a_{P_\alpha}\| \leq \|a\|.$$

Hence $(\lambda(a_{P_\alpha}))$ lies in a compact subset of the complex plane and by [21], Chapter 5, Theorem 2, has a subnet $\left(\lambda\left(a_{P_\beta}\right)\right)_{\beta\in\mathcal{B}}$ convergent to some complex number μ. Let a be an element of A. As we have seen, there exists a pure functional f on A/I such that

$$|f(a_I - \mu)| = \|a_I - \mu\|$$

and f induces a pure functional f on A such that $\Gamma_A(f)$ lies in $\mathrm{hull}\, I$ and is therefore a limit of (P_β). Since Γ_A is open and surjective, by Lemma 4.3 there exists a subnet $(P_\omega)_{\omega\in\Omega}$ and a net $(f_\omega)_{\omega\in\Omega}$ weak* convergent to f such that $\Gamma_A(f_\omega)$ is P_ω for all ω in Ω. Then, given $\varepsilon > 0$, there exists a ω in Ω such that

$$|\lambda(a_{P_\omega}) - \mu| < \tfrac{\varepsilon}{2}, \qquad |f(a-\mu) - f_\omega(a-\mu)| < \tfrac{\varepsilon}{2}.$$

Recalling that f_ω induces a pure functional on A/P_ω, we have

$$\begin{aligned}\|a_{P_\omega} - \lambda(a_{P_\omega})\| &\geq |f_\omega(a_{P_\omega} - \lambda(a_{P_\omega}))| \\ &\geq |f_w(a-\mu)| - |f_\omega(\lambda(a_{P_\omega}) - \mu)| \\ &\geq |f(a-\mu)| - |f_w(a-\mu) - f(a-\mu)| - \tfrac{\varepsilon}{2} \\ &\geq |f(a-\mu)| - \tfrac{\varepsilon}{2} - \tfrac{\varepsilon}{2} \\ &= \|a_I - \mu\| - \varepsilon \\ &\geq \|a_I - \lambda(a_I)\| - \varepsilon.\end{aligned}$$

Thus

$$\sup\{\|a_P - \lambda(a_P)\| : P \in \mathrm{Prim}\, A\} \geq \|a_I - \lambda(a_I)\|$$

for any primal ideal I of A. Hence

$$\sup\{\|a_P - \lambda(a_P)\| : P \in \mathrm{Prim}\, A\} = \sup\{\|a_P - \lambda(a_P)\| : P \in \mathrm{Primal}\, A\}$$

and this equals $\|\mathrm{ad}_A\|$ by Corollary 3.9. □

For a unital C*-algebra A, let

$$N_A = \overline{\mathrm{conv}\{f \in G_A : f(1) \geq 0\}}.$$

For a in A, let

$$U_A(a) = \{f(a) : f \in N_A\}.$$

Since $S_A^0(a)$ is a non-empty weak*-compact face of S_A, by the Krein-Milman Theorem, there exists an element x in $\partial_e S_A$ such that $x(a^*a) = \|a\|^2$. Define $f : A \to \mathbb{C}$ by

$$f(b) = \langle \pi(b)\xi_x, \eta\rangle$$

where

$$\eta = e^{i\theta} \frac{\pi(a)}{\|a\|} \xi_x$$

and θ is an element of $[0, 2\pi]$, chosen so that $e^{i\theta} x(a) \geq 0$. Since

$$\|\pi(a)\xi_x\|^2 = \langle \pi(a^*a)\xi_x, \xi_x \rangle = x(a^*a) = \|a\|^2$$

η is a unit vector and f lies in G_A. Since

$$f(b) = \frac{e^{-i\theta}}{\|a\|} x(a^*b),$$

it follows that $f(1)$ is positive by choice of θ, and that $f(a) = e^{i\theta}\|a\|$. Thus, for all a in A, there exists f in N_A such that $|f(a)| = \|a\|$ and hence there exists an element of norm $\|a\|$ in $U_A(a)$. Thus,

$$\|a\| = \sup\{|f(a)| : f \in N_A\}.$$

In Chapter 3 the circumcircle was defined and used to find an expression for $\lambda(a)$ when a is self-adjoint. We now prove some inequalities connecting $\lambda(a)$ and the circumcentre of $U_A(a)$ for a general element a of A.

Lemma 4.5. *Let A be a unital C*-algebra. Let a be an element of A, let $\mu(a)$ denote the circumcentre of $U_A(a)$ and let $\rho(a)$ denote the circumradius of $U_A(a)$. Then:*

(i) $|\lambda(a)|^2 \leq 4|\mu(a)|\,\|a\|$;

(ii) *if* $|\mu(a)| \leq \frac{1}{2}\|a\|$ *then* $|\lambda(a)|^2 \leq 4|\mu(a)|(\|a\| - |\mu(a)|)$;

(iii) $\|a\|^2 \geq \rho(a)^2 + |\mu(a)|^2$;

(iv) $|\lambda(a) - \mu(a)|^2 \leq 2|\lambda(a)|\,\|a\|$.

Proof. Fix a in A and let $\mu = \mu(a)$, $\lambda = \lambda(a)$ and $\rho = \rho(a)$. As shown above, there exists α in $U_A(a)$ such that $|\alpha| = \|a\|$. Therefore

$$\rho \geq |\alpha - \mu| \geq \|a\| - |\mu|.$$

The diameter perpendicular to the line $\overrightarrow{\lambda\mu}$ divides the circumcircle into two semicircles. By property (i) of circumcircles there exists f in N_A such that $\alpha = f(a)$ is in the closed semicircle not cut by $\overrightarrow{\lambda\mu}$. Let θ be the angle between $\overrightarrow{\lambda\mu}$ and $\overrightarrow{\mu\alpha}$. By construction $\rho = |\alpha - \mu|$ and θ lies in $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, so $\cos\theta \leq 1$. Let t equal $f(1)$. Since t lies in $[0, 1]$, the complex number β, given by $t\lambda + (1-t)\mu$, lies on $\overrightarrow{\lambda\mu}$.

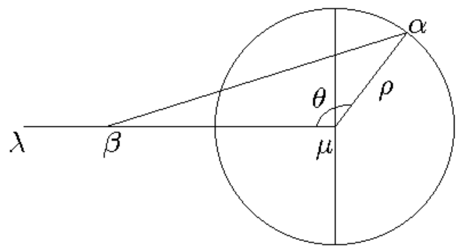

Figure 4.1: The angle θ.

The cosine rule and $\cos\theta \leq 1$ gives the inequality

$$|\alpha - \beta| \geq |\alpha - \mu| = \rho.$$

Now,

$$\begin{aligned}\|a-\lambda\| &\geq |f(a-\lambda)| \\ &= |\alpha - t\lambda| \\ &\geq ||\alpha - t\lambda - (1-t)\mu| - |(1-t)\mu|| \\ &\geq |\alpha-\beta| - (1-t)|\mu| \\ &\geq \rho - |\mu| \\ &\geq \|a\| - 2|\mu|,\end{aligned}$$

from which it follows that

$$2|\mu| \geq \|a\| - \|a-\lambda\|, \qquad -\|a-\lambda\|^2 \leq -(\|a\| - 2|\mu|)^2.$$

By Theorem 3.4

$$\|a-\lambda\|^2 + |\lambda|^2 \leq \|a\|^2, \qquad \|a-\lambda\| \leq \|a\|.$$

Thus,

$$\begin{aligned}|\lambda|^2 &\leq \|a\|^2 - \|a-\lambda\|^2 \\ &= (\|a\| - \|a-\lambda\|)(\|a\| + \|a-\lambda\|) \\ &\leq 2|\mu| \, 2\|a\| \\ &= 4|\mu| \, \|a\|.\end{aligned}$$

If $|\mu(a)| \leq \frac{1}{2}\|a\|$, then

$$|\lambda|^2 \leq \|a\|^2 - \|a-\lambda\|^2 \leq \|a\|^2 - (\|a\| - 2|\mu|)^2 = 4|\mu|(\|a\| - |\mu|).$$

This proves (1) and (2). By the method used above we may choose f in N_A such that the angle between μ and $\overrightarrow{\mu\alpha}$ lies in $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ where $\alpha = f(a)$. By the cosine rule

$$\|a\|^2 \geq |f(a)|^2 \geq \rho^2 + |\mu|^2.$$

This proves (3). From (3), or the observation that $U_A(a)$ is a subset of $B_{\|a\|}(0)$, it can be seen that $\|a\| \geq \rho$. Let f be an element of N_A such that $|f(a-\mu)| > \rho$. Then $f(1)$ lies in $[0,1)$ since $f(1) = 1$ implies

$$|f(a-\mu)| = |f(a) - \mu| \leq \rho.$$

Let

$$\alpha = f(a), \qquad \beta = f(1)\mu,$$

and let ϕ, θ be the angles indicated in Figure 4.2. Then,

$$|\alpha - \beta| = |f(a-\mu)| > \rho \geq |f(a) - \mu|$$

and, by the cosine rule, $\cos\phi \geq 0$. Hence $\cos\theta \leq 0$. Applying the cosine rule again gives that

$$\|a\| \geq |\alpha| \geq |\beta - \alpha| = |f(a-\mu)|.$$

Thus, for all f in N_A, either

$$|f(a-\mu)| \leq \rho \leq \|a\|,$$

or $|f(a-\mu)| > \rho$, in which case $|f(a-\mu)| \leq \|a\|$. Therefore,

$$\|a\| \geq \sup\{|f(a-\mu)| : f \in N_A\} = \|a-\mu\|.$$

By Theorem 3.4,

$$\begin{aligned}|\lambda-\mu|^2 &\leq \|a-\mu\|^2 - \|a-\lambda\|^2 \\ &\leq \|a\|^2 - \|a-\lambda\|^2 \\ &= (\|a\| - \|a-\lambda\|)(\|a\| + \|a-\lambda\|) \\ &\leq 2|\lambda| \, \|a\|\end{aligned}$$

and the proof of (4) is complete. □

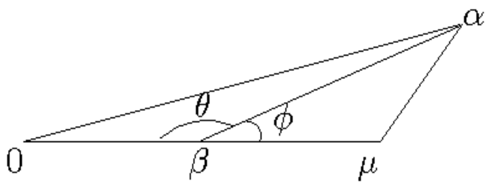

Figure 4.2: The angles θ and ϕ.

Chapter 5

The Categorisation Theorem

In this chapter we develop the main theorem of this paper, the categorisation theorem given in [31], Section 3. This is done by proving four theorems connecting the values of $K_s(A)$ and $K(A)$ to the primality of the intersection of a fixed number of primitive ideals containing the same Glimm ideal.

We begin by quoting two technical results which will be used in the following theorems. They can be found in [2], Theorem 4.3 and [1], Corollary 2.4 respectively.

Proposition 5.1. *Let π be a *-homomorphism on a C*-algebra A. Then, for each element a in A, there exists b in $\ker \pi$ such that b is an element of $C^*(a)$, the smallest C*-subalgebra of A containing a, and*

$$\|a-b\| = \|\pi(a)\|.$$

Since every ideal I of A is the kernel of the quotient homomorphism, an immediate corollary of Proposition 5.1 is that, for all a in A, there exists b in I such that

$$\|a-b\| = \|a_I\|.$$

Proposition 5.2. *Let A be a C*-algebra and let a and b be elements of A_{sa} such that ab lies in I. Then there exist a_1 and b_1 in I such that $a-a_1$, $b-b_1$ lie in I_{sa} and $(a-a_1)(b-b_1)$ is zero.*

Note that since a and b lie in A_{sa} and $a-a_1$ and $b-b_1$ lie in I_{sa} in the statement of Proposition 5.2, a_1 and b_1 lie in I_{sa}.

The first theorem, a reformulation of [30], Theorem 5.8, gives necessary and sufficient conditions for $K_s(A)$ to equal $\frac{1}{2}$. It also shows that, for a non-commutative C*-algebra, if $K_s(A)$ is not equal to $\frac{1}{2}$, $K(A) \geq K_s(A) \geq 1$. The proof given here follows [30], Theorem 5.8, but it is also possible to deduce the result from [32], Theorem 4.4 (see Chapter 6).

Theorem 5.3. *Let A be a non-commutative unital C*-algebra. If, whenever two primitive ideals of A contain the same Glimm ideal, their intersection is primal, then $K_s(A) = \frac{1}{2}$. Otherwise $K_s(A) \geq 1$.*

Proof. Suppose that whenever two primitive ideals of A contain the same Glimm ideal their intersection is primal. Let G be a Glimm ideal and let a be a self adjoint element of A. As we have seen, there exist primitive ideals P and Q of A containing G, such that

$$\alpha(a_G) = \alpha(a_R), \qquad \beta(a_G) = \beta(a_R),$$

where R is the intersection of P and Q. Since, by hypothesis, R is primal, by Theorem 4.4,

$$\|a_R - \lambda(a_R)\| \leq \tfrac{1}{2}\|\mathrm{ad}_A\, a\|.$$

Since a is self adjoint,

$$\begin{aligned}\|a_R - \lambda(a_R)\| &= \tfrac{1}{2}(\alpha(a_R) - \beta(a_R))\\ &= \tfrac{1}{2}(\alpha(a_G) - \beta(a_G))\\ &= \|a_G - \lambda(a_G)\|.\end{aligned}$$

Since G was arbitrary, this implies that

$$\sup \{\|a_G - \lambda (a_G)\| : G \in \mathrm{Glimm}\, A\} \leq \tfrac{1}{2} \|\mathrm{ad}_A\, a\| .$$

Then, by Theorem 3.11,

$$d(a, Z(A)) \leq \tfrac{1}{2} \|\mathrm{ad}_A\, a\|$$

for all a in A_{sa}. Therefore $K_s(A) \leq \frac{1}{2}$. Since A is non-commutative, $K_s(A) \geq \frac{1}{2}$, and consequently, $K_s(A) = \frac{1}{2}$.

Conversely, suppose there exist primitive ideals P and Q of A such that their intersection contains the same Glimm ideal, but is not primal. Then, by Corollary 2.8, there exist ideals I and J of A such that

$$I \nsubseteq P, \qquad J \nsubseteq Q, \qquad IJ = \{0\} .$$

We may choose a in I but not P. By Proposition 5.1, there exists b in $G \cap C^*(a)$ such that $\|c\| = \|c_G\|$ where $c = a - b$. Since $C^*(a)$ is a subset of I, c is an element of I but not of G. Since zero is an element of G, c is non-zero, and we may rescale c to have unit norm. Then c^*c is a positive element of I and

$$\|c^*c\| = \|(c^*c)_G\| = 1.$$

By this argument we may choose b in $I^+ \backslash G$ and c in $J^+ \backslash Q$ such that

$$\|b\| = \|b_G\| = \|c\| = \|c_G\| = 1.$$

Furthermore, since IJ is zero, bc equals zero. Let $a = b - c$. Then a_G is self-adjoint, and by the uniqueness of the orthogonal decomposition,

$$b_G = a_G^+, \qquad c_G = a_G^-.$$

As we have seen, in these circumstances $a_G - \lambda(a_G)$ has unit norm, so by Theorem 3.11

$$d(a, Z(A)) \geq \|a_G - \lambda (a_G)\| = 1.$$

Let R be a primitive ideal. Since IJ is zero, R contains at least one of I and J. Hence R contains at least one of a^+ and a^-. Therefore, for each R in $\mathrm{Prim}\, A$, a_R is either a_R^+ or a_R^- . Since a_R^+ is positive

$$\begin{aligned} \left\|a_R^+ - \lambda\left(a_R^+\right)\right\| &= \tfrac{1}{2}\left(\alpha\left(a_R^+\right) - \beta\left(a_R^+\right)\right) \\ &\leq \tfrac{1}{2}\left\|a_R^+\right\| \\ &\leq \tfrac{1}{2}. \end{aligned}$$

Similarly for a_R^-. Therefore

$$2 \sup \{\|a_R - \lambda (a_R)\| : R \in \mathrm{Prim}\, A\} \leq 1.$$

Then, by Theorem 4.4,

$$\|\mathrm{ad}_A\, a\| \leq 1 \leq d(a, Z(A)) \leq K_s(A) \|\mathrm{ad}_A\, a\| .$$

Since $d(a, Z(A))$ is non-zero, $\|\mathrm{ad}_A\, a\|$ is non-zero, and this implies that $K_s(A) \geq 1$. ☐

Theorem 5.4. *Let A be a non-commutative unital C^*-algebra such that, whenever three primitive ideals of A contain the same Glimm ideal, their intersection is primal. Then $K(A) = \frac{1}{2}$.*

Proof. Let a be an element of A and let G be an element of $\mathrm{Glimm}\, A$. Let $b = a - \lambda(a_G)$. Then $\lambda(b_G)$ is zero by Theorem 3.4. Let B be the C*-algebra A/G, let λ be an element of $\partial_e U_B(b_G)$ and define

$$F = \{f \in N_B : f(b_G) = \lambda\} .$$

By definition of $U_B(b_G)$, F is non-empty. F is clearly convex, and since λ is extreme in $U_B(b_G)$, F is a face of N_B. Furthermore

$$F = N_B \cap \widehat{b_G}^{-1}(\{\lambda\}).$$

Thus F is weak*-closed in the weak*-compact set A_1^* and is hence weak*-compact. Hence the Krein-Milman theorem applies to F, and there exists f in $\partial_e N_B$ such that $f(b_G) = \lambda$. By property (ii) of circumcircles there exist points $f(b_G)$, $g(b_G)$, $h(b_G)$ with the same circumcircle as $U_B(b_G)$ for some f, g, h in $\partial_e N_B$. Since $\overline{\{g \in G_B : g(1) \geq 0\}}$ is a closed subset of N_B such that

$$N_B = \overline{\operatorname{conv}\overline{\{g \in G_B : g(1) \geq 0\}}},$$

it follows from the Krein-Milman theorem that

$$\partial_e N_B \subseteq \overline{\{g \in G_B : g(1) \geq 0\}}.$$

Hence there exist nets $(f_\alpha)_{\alpha\in\mathcal{A}}$, $(g_\beta)_{\beta\in\mathcal{B}}$, $(h_\gamma)_{\gamma\in\mathcal{C}}$ in G_B convergent to f, g, h respectively. Let $\Lambda = \mathcal{A} \times \mathcal{B} \times \mathcal{C}$ have the product direction and define

$$f_{(\alpha,\beta,\gamma)} = f_\alpha, \qquad g_{(\alpha,\beta,\gamma)} = g_\beta, \qquad h_{(\alpha,\beta,\gamma)} = h_\gamma.$$

Then it is easy to check that $(f_\lambda)_{\lambda\in\Lambda}$, $(g_\lambda)_{\lambda\in\Lambda}$ and $(h_\lambda)_{\lambda\in\Lambda}$ are nets convergent to f, g, h respectively.

For each λ in Λ, define primitive ideals P_λ, Q_λ and R_λ of A by

$$P_\lambda = \Gamma(f_\lambda \circ p_G), \qquad Q_\lambda = \Gamma(g_\lambda \circ p_G), \qquad R_\lambda = \Gamma(h_\lambda \circ p_G).$$

Let S_λ be the ideal $P_\lambda \cap Q_\lambda \cap R_\lambda$. Clearly G is a subset of S_λ, and by hypothesis S_λ is primal.

Since S_λ is a subset of $\ker f_\lambda \circ p_G$, a well defined function is induced on A/S_λ by f_λ. Let ρ_λ be the circumradius of $U_{A/S_\lambda}(b_{S_\lambda})$, and let $\varepsilon > 0$. Since f is the weak*-limit of (f_λ), there exists λ_0 in Λ such that $|f_\lambda(b_G) - f(b_G)| < \varepsilon$ for $\lambda \geq \lambda_0$. Hence

$$|\mu(b_{S_\lambda}) - f(b_G)| \leq |\mu(b_{S_\lambda}) - f_\lambda(b_{S_\lambda})| + |f_\lambda(b_{S_\lambda}) - f(b_G)| \leq \rho_\lambda + \varepsilon$$

for all $\lambda \geq \lambda_0$. Similar inequalities hold for g and h. Thus, the circle with centre $\mu(b_{S_\lambda})$ and radius $\rho_\lambda + \varepsilon$ contains the points $f(b_G)$, $g(b_G)$, $h(b_G)$, and hence $\rho_\lambda + \varepsilon$ is larger than the radius of their circumcircle, which by construction is the circumcircle of $U_B(b_G)$. Since $\lambda(b_G)$ equals zero, it follows from
Lemma 4.5 (4) that $\mu(b_G)$ is zero. Combining this with the fact that $U_B(b_G)$ has an element of norm $\|b_G\|$ it can be seen that, for λ sufficiently large,

$$\rho_\lambda \leq \|b_{S_\lambda}\| \leq \|b_G\| \leq \rho_\lambda + \varepsilon.$$

Since ε was arbitrary, it follows that ρ_λ converges to $\|b_G\|$. Applying this to Lemma 4.5 (3) shows that

$$\rho_\lambda^2 + |\mu(b_{S_\lambda})|^2 \leq \|b_{S_\lambda}\|^2 \leq \|b_G\|^2$$

from which it follows that $\mu(b_{S_\lambda})$ converges to zero. Applying this to Lemma 4.5 (1) shows that $\lambda(b_{S_\lambda})$ converges to zero. Thus, given $\varepsilon > 0$, there exists a λ such that

$$|\lambda(b_{S_\lambda})| < \tfrac{\varepsilon}{4}, \qquad |f_\lambda(b_G) - f(b_G)| < \tfrac{\varepsilon}{4}.$$

Furthermore,

$$\begin{aligned}
\|b_{S_\lambda} - \lambda(b_{S_\lambda})\| &\geq |f_\lambda(b_{S_\lambda} - \lambda(b_{S_\lambda}))| \\
&\geq |f_\lambda(b_{S_\lambda})| - |\lambda(b_{S_\lambda})|\,|f_\lambda(1)| \\
&\geq |f(b_G)| - |f_\lambda(b_{S_\lambda}) - f(b_G)| - |\lambda(b_{S_\lambda})| \\
&\geq |f(b_G)| - \tfrac{\varepsilon}{4} - \tfrac{\varepsilon}{4}.
\end{aligned}$$

Since $\mu(b_G)$ is zero and $U_A(b_G)$ has an element of norm $\|b_G\|$, the circumradius of $U_B(b_G)$ is $\|b_G\|$. The points $f(b_G)$, $g(b_G)$, $h(b_G)$ lie on the circumference of this circle and therefore have magnitude $\|b_G\|$. Since $\lambda(b_G)$ is zero, for any Glimm ideal G and any $\varepsilon > 0$, there exists an element S in Primal A such that

$$2\|b_S - \lambda(b_S)\| \geq 2\|b_G\| - \varepsilon = 2\|b_G - \lambda(b_G)\| - \varepsilon.$$

By Proposition 4.4 it follows that, for all G in Glimm A,

$$\|ad_A b\| \geq 2\|b_G - \lambda(b_G)\| - \varepsilon$$

and, by Theorem 3.11,

$$\|ad_A b\| \geq 2d(b, Z(A)) - \varepsilon$$

for all $\varepsilon > 0$. Thus,

$$\|ad_A a\| = \|ad_A b\| \geq 2d(b, Z(A)) = 2d(a, Z(A)),$$

and $K(A) \leq \frac{1}{2}$. Since A is non-commutative $K(A) \geq \frac{1}{2}$ and the theorem is proved. □

The following theorem is [31], Theorem 3.3.

Theorem 5.5. *Let A be a unital C^*-algebra with three primitive ideals containing the same Glimm ideal which have non-primal intersection. Then $K(A) \geq \frac{1}{\sqrt{3}}$.*

Proof. Let P, Q and R be primitive ideals containing the same Glimm ideal G such that $P \cap Q \cap R$ is non-primal. As shown in Theorem 5.3 above, if A has two primitive ideals which contain the same Glimm ideal and whose intersection is non-primal then $K(A) \geq K_S(A) \geq 1$ and there is nothing more to prove. Therefore we may assume $P \cap Q$, $Q \cap R$ and $R \cap P$ are primal.

Since $P \cap Q \cap R$ is not primal, by Corollary 2.8 there exist ideals I, J and K of A such that

$$I \nsubseteq P, \qquad J \nsubseteq Q, \qquad K \nsubseteq R, \qquad IJK = \{0\}.$$

Since $P \cap Q$ is primal, applying Corollary 2.8 again gives that IJ is non-zero. Similarly, JK and KI are non-zero. Since IJK is zero, $Q \cap R$ is primal, and K is not a subset of R, it follows that IJ is a subset of $Q \cap R$ and hence of Q. Since Q is primitive, but does not contain J, I is a subset of Q. Similarly,

$$I \subseteq Q, R, \qquad J \subseteq P, R, \qquad K \subseteq P, Q.$$

Thus,

$$P \supseteq J + K, \qquad Q \supseteq I + K, \qquad R \supseteq J + I.$$

Arguing as in Theorem 5.3, we may choose a in $I^+ \backslash P$, b in $J^+ \backslash Q$ and c in $K^+ \backslash R$ such that

$$\|a\| = \|a_P\| = \|b\| = \|b_P\| = \|c\| = \|c_P\| = 1$$

Then a, b and c are self adjoint and ab lies in IJ, bc lies in JK and ca lies in KI. Hence, by Proposition 5.2, there exist a_1, b_1 in IJ_{sa}, b_2, c_2 in JK_{sa} and c_3, a_3 in KI_{sa} such that

$$(a - a_1)(b - b_1) = (b - b_2)(c - c_2) = (c - c_3)(a - a_3) = 0.$$

Define self-adjoint elements d, f and g of A by

$$d = a - a_1 - a_3, \qquad f = b - b_1 - b_2, \qquad g = c - c_2 - c_3.$$

Since $a_1 + a_3$ is an element of $J + K$, and hence of P, it follows that d_P equals a_P. Similarly f_Q equals b_Q and g_R equals c_R. Now,

$$\begin{aligned} df &= (a - a_1 - a_3)(b - b_1 - b_2) \\ &= (a - a_1)(b - b_1) - (a - a_1) b_2 - a_3 (b - b_1) + a_3 b_2 \\ &= 0, \end{aligned}$$

since $(a - a_1)(b - b_1)$ is zero and the other terms are elements of IJK. Similarly, fg and gd are zero. By Proposition 5.1, there exists p in $P \cap C^*(d)$ such that

$$\|d - p\| = \|d_P\| = \|a_P\| = 1.$$

Since every element of $C^*(d)$ is an element of $P \cap I$ and has zero product with f and g, we may replace d with $(d-p)^2$. Doing the same to f and g shows that A has positive elements d, f and g such that:

(i) d lies in I, f in J and g in K;

(ii) df, fg and gd are zero;

(iii) d, f, g, d_P, f_Q and g_R have unit norm.

Define an element h of A by

$$h = d + e^{\frac{2\pi i}{3}} f + e^{\frac{4\pi i}{3}} g.$$

Then h is normal and the decomposition is the unique decomposition of h discussed in Chapter 3. As shown there, h has unit norm. Since G is contained in P, Q and R, it follows that

$$1 = \|d_P\| \leq \|d_G\| \leq \|d\| = 1.$$

Similarly f_G and g_G have unit norm. Therefore,

$$h_G = d_G + e^{\frac{2\pi i}{3}} f_G + e^{\frac{4\pi i}{3}} g_G$$

is the decomposition of h_G, h_G is of unit norm and $d(h_G, \mathbb{C}1_G)$ is 1. Using Theorem 3.11,

$$1 = \|h\| \geq d(h, Z(A)) \geq d(h_G, \mathbb{C}1_G) = 1.$$

Thus $d(h, Z(A))$ is 1.

Let S be a primitive ideal of A. Since d, f and g have unit norm, d_S, f_S and g_S lie in the unit ball of A/S. Since IJK is zero, S contains at least one of d, f and g. Suppose d lies in S. Then d_S is zero and, as argued in Chapter 3,

$$\|h_S - \lambda(h_S)\| = d(h_S, \mathbb{C}1_S) \leq \tfrac{\sqrt{3}}{2}.$$

Similarly if f or g lie in S. By Corollary 3.9,

$$\|\mathrm{ad}_A\, a\| \leq \sqrt{3} = \sqrt{3} d(h, Z(A)) \leq \sqrt{3} K(A) \|\mathrm{ad}_A\, a\|.$$

Therefore $K(A) \geq \frac{1}{\sqrt{3}}$. □

The last component of the categorisation theorem is [31], Theorem 3.4.

Theorem 5.6. *Let A be a unital C*-algebra such that $K_s(A) = \frac{1}{2}$. Then $K(A) \leq \frac{1}{\sqrt{3}}$.*

Proof. Let a be an element of A and let G be a Glimm ideal of A. Let b be the element $a - \lambda(a_G)$ of A and let B be the C*-algebra A/G. Then $\lambda(b_G)$ is zero by Theorem 3.4. By Lemma 3.2 (4), $U_A(b)$ has its circumcentre at the origin and it follows that the circumradius is $\|b\|$. Lemma 3.5 implies that there exist elements f and g of N_B such that

$$|f(b_G) - g(b_G)| \geq \sqrt{3}\|b_G\|, \qquad |f(b_G)| = |g(b_G)| = \|b_G\|.$$

As in the proof of Theorem 5.4, there exist nets $(f_\lambda)_{\lambda \in \Lambda}$ and $(g_\lambda)_{\lambda \in \Lambda}$ in the set $\{g \in G_B : g(1) \geq 0\}$ converging to f and g respectively with corresponding families of primitive ideals

$$P_\lambda = \Gamma_A(f_\lambda \circ p_G), \qquad Q_\lambda = \Gamma_A(g_\lambda \circ p_G)$$

containing G. Let R_λ be the intersection of P_λ and Q_λ. Since $K_s(A)$ is $\frac{1}{2}$, Theorem 5.3 implies that R_λ is primal.

Since $(f_\lambda)_{\lambda\in\Lambda}$ and $(g_\lambda)_{\lambda\in\Lambda}$ are weak* convergent to f and g respectively, for each $\varepsilon > 0$ there exists λ in Λ such that

$$|f_\lambda(b_G) - f(b_G)| < \varepsilon, \qquad |g_\lambda(b_G) - g(b_G)| < \varepsilon.$$

Then

$$|f_\lambda(b_G) - g_\lambda(b_G)| \geq \sqrt{3}\,\|b_G\| - 2\varepsilon.$$

Let L be the perpendicular bisector of the line from $f_\lambda(b_G)$ to $g_\lambda(b_G)$. Since $f_\lambda(b_G)$ and $g_\lambda(b_G)$ are the same distance from the origin, L passes through the origin. Let μ be a complex number. Since $f_\lambda(1)$ and $g_\lambda(1)$ are both positive, $\mu f_\lambda(1)$ and $\mu g_\lambda(1)$ both lie on the same side of L, see Figure 5.1. Hence at least one of the inequalities

$$|f_\alpha(b_G - \mu)| \geq \tfrac{1}{2}\sqrt{3}\,\|b_G\| - \varepsilon$$
$$|g_\alpha(b_G - \mu)| \geq \tfrac{1}{2}\sqrt{3}\,\|b_G\| - \varepsilon$$

must hold. Thus,

$$\begin{aligned}\|b_{R_\lambda} - \lambda(b_{R_\lambda})\| &\geq \max\{|f_\alpha(b_{R_\lambda} - \lambda(b_{R_\lambda}))|, |g_\alpha(b_{R_\lambda} - \lambda(b_{R_\lambda}))|\} \\ &\geq \tfrac{1}{2}\sqrt{3}\,\|b_G\| - \varepsilon.\end{aligned}$$

It now follows that

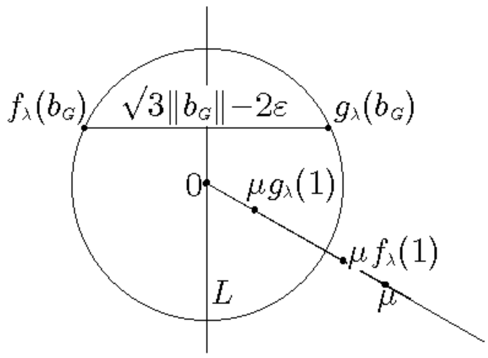

Figure 5.1: The positions of $\mu f_\lambda(1)$ and $\mu g_\lambda(1)$.

$$\begin{aligned}\|\mathrm{ad}_A\, a\| &= \|\mathrm{ad}_A\, b\| \\ &= 2\sup\{\|b_P - \lambda(b_P)\| : P \in \mathrm{Primal}\, A\} \\ &\geq \sqrt{3}\,\|b_G\| \\ &= \sqrt{3}\,\|a_G - \lambda(a_G)\|\,.\end{aligned}$$

Since G was arbitrary, it follows that

$$d(a, Z(A)) \leq \tfrac{1}{\sqrt{3}}\,\|\mathrm{ad}_A\, a\|\,,$$

and hence $K(A) \leq \frac{1}{\sqrt{3}}$. □

We are now in a position to state and prove the categorisation theorem [31], Section 3.

Theorem 5.7. *Let A be a unital C^*-algebra. Then $K(A)$ and $K_s(A)$ are zero if and only if A is commutative. When A is non-commutative there are three mutually exclusive cases:*

(i) $K(A) = K_s(A) = \frac{1}{2}$ *if and only if whenever three primitive ideals of A contain the same Glimm ideal of A their intersection is primal.*

(ii) $K(A) = 1/\sqrt{3}$, $K_s(A) = \frac{1}{2}$ *if and only if whenever two primitive ideals of A contain the same Glimm ideal of A their intersection is primal, but there exist three primitive ideals containing the same Glimm ideal whose intersection is not primal.*

(iii) $K(A) \geq K_s(A) \geq 1$ *if and only if A has two primitive ideals which contain the same Glimm ideal, but whose intersection is not primal.*

Proof. The commutative case is immediate from the definition. Consider A non-commutative. Suppose that whenever three primitive ideals contain the same Glimm ideal their intersection is primal. By Theorem 5.4, $K(A) = \frac{1}{2}$. Since A is non-commutative, $K(A) \geq K_s(A) \geq \frac{1}{2}$ so $K_s(A) = \frac{1}{2}$. The converse of (1) follows from Theorem 5.5.

Suppose that whenever two primitive ideals contain the same Glimm ideal their intersection is primal, but there exist three primitive ideals containing the same Glimm ideal whose intersection is not primal. Then, by Theorem 5.3, $K_s(A) = \frac{1}{2}$ and, by Theorem 5.5 and Theorem 5.6, $K(A) = 1/\sqrt{3}$. Conversely, if $K(A) = 1/\sqrt{3}$ and $K_s(A) = \frac{1}{2}$, then every pair of primitive ideals containing the same Glimm ideal has primitive intersection by Theorem 5.3, but there exist three primitive ideals containing the same Glimm ideal with non-primal intersection to avoid contradicting Theorem 5.4. This proves (2).

Finally, (3) follows immediately from Theorem 5.3. □

We now give some examples to show that each of the possibilities in Theorem 5.7 occur.

Example 5.8. Let H be a separable infinite-dimensional Hilbert space. As shown in [20], page 747, the only ideals of $B(H)$ are $\{0\}$, $K(H)$ and $B(H)$ where $K(H)$ is the C*-algebra of compact linear operators. The identity map on $B(H)$ is an irreducible representation so $\{0\}$ is a primitive ideal The hull of $K(H)$ is non-empty, but the only possible element is $K(H)$, so $K(H)$ is a primitive ideal. Hence

$$\operatorname{Prim} B(H) = \{\{0\}, K(H)\}.$$

Since $\operatorname{Prim} B(H)$ is closed under intersection and primitive ideals are primal, it follows that every intersection of primitive ideals is primal. Thus $K(A)$ and $K_s(A)$ are $\frac{1}{2}$. This agrees with the results of [33] as described in Chapter 3.

The following example is given in [31], Example 3.5.

Example 5.9. Let A be the set of sequences (a_n) in $M_2(\mathbb{C})$ such that the subsequences (a_{3r}), (a_{3r+1}) and (a_{3r+2}) converge in $M_2(\mathbb{C})$ to matrices of the form

$$\begin{pmatrix} \lambda_1(a) & 0 \\ 0 & \lambda_2(a) \end{pmatrix}, \qquad \begin{pmatrix} \lambda_2(a) & 0 \\ 0 & \lambda_0(a) \end{pmatrix}, \qquad \begin{pmatrix} \lambda_0(a) & 0 \\ 0 & \lambda_1(a) \end{pmatrix},$$

for some complex numbers $\lambda_0(a)$, $\lambda_1(a)$ and $\lambda_2(a)$. With pointwise defined operations and the supremum norm, A is a C*-algebra. Define a *-homomorphism $\pi_n : A \to M_2(\mathbb{C})$ to be the natural map giving the n^{th} term of A. Clearly π_n is surjective and it is elementary to check that $M_2(\mathbb{C})'$ is $\mathbb{C}1$. Hence π_n is an irreducible representation of A and P_n is a primitive ideal where

$$P_n = \ker \pi_n = \{a \in A : a_n = 0\} \qquad n \in \mathbb{N}.$$

Define three characters on A by

$$x_j(a) = \lambda_j(a) \qquad j = 0, 1, 2.$$

Then Q_0, Q_1 and Q_2 are also primitive ideals of A where

$$Q_j = \ker x_j = \{a \in A : \lambda_j(a) = 0\} \qquad j = 0, 1, 2.$$

It can be shown that these are all the primitive ideals of A. Recall from our discussion of Glimm ideals that two primitive ideals contain the same Glimm ideal if and only if their intersections with the centre agree. It is easy to check that

$$Z(A) = \{(z_n) \in A : z_n \in Z(M_2(\mathbb{C}))\ \forall n \in \mathbb{N}\},$$

and that

$$P_n \cap Z(A) = P_m \cap Z(A) \quad \Leftrightarrow \quad n = m.$$

Similarly,

$$P_n \cap Z(A) \neq Q_j \cap Z(A) \qquad j = 0, 1, 2$$

so P_n is the only element of $[P_n]$, and hence the P_n are all Glimm ideals. Recall that, for all a in $M_2(\mathbb{C})$,

$$\begin{aligned} \|a\| &= \sup\left\{\|ax\| : x \in \mathbb{C}^2,\ \|x\| = 1\right\} \\ &\geq \sqrt{|a_{11}|^2 + |a_{21}|^2},\ \sqrt{|a_{12}|^2 + |a_{22}|^2} \\ &\geq |a_{11}|,\ |a_{21}|,\ |a_{12}|,\ |a_{22}|. \end{aligned}$$

Let (a_n) be an element of A. Then

$$\left\| a_{3n} - \begin{pmatrix} \lambda_1(a) & 0 \\ 0 & \lambda_2(a) \end{pmatrix} \right\| \to 0 \text{ as n} \to \infty$$

so $\left(a_{11}^{3n}\right)$ converges to $\lambda_1(a)$ and $\left(a_{22}^{3n}\right)$ converges to $\lambda_2(a)$. Let U be an open neighbourhood of Q_1 and let C be the closed set $\operatorname{Prim} A \backslash U$. Suppose that, for all natural numbers n, there exists N_n greater than n such that P_{3N_n} lies in C. Then (P_{3N_n}) is a sequence in C and $\ker C$ is a subset of I, where

$$I = \bigcap_{n \in \mathbb{N}} P_{3N_n}.$$

Let (a_n) be an element of I. Then a_{3N_n} is zero for each n in $\mathbb{N}$. Thus

$$\lambda_1(a) = \lim_{n \to \infty} \left(a_{11}^{3n}\right) = \lim_{n \to \infty} \left(a_{11}^{3N_n}\right) = 0.$$

This implies that I is a subset of Q_1 and hence that Q_1 lies in C which is a contradiction. Therefore there exists n_0 in $\mathbb{N}$ such that P_{3N_n} lies in U for all n greater than n_0. We have shown that P_{3N_n} is a net in $\operatorname{Prim} A$ converging to Q_1. Similarly P_{3N_n} converges to Q_2. It is clear that $\{Q_1, Q_2\}$ is the hull of $Q_1 \cap Q_2$ so it follows from Theorem 2.7 that $Q_1 \cap Q_2$ is primal. Similarly $Q_2 \cap Q_0$ and $Q_0 \cap Q_1$ are primal. Hence Q_0, Q_1 and Q_2 all belong to the same equivalence class and are the only ideals in this class. The only distinct primitive ideals which contain the same Glimm ideal are Q_0, Q_1 and Q_2. As we have seen any two of these have primal intersection. Define

$$I_j = \cap_{n \in \mathbb{N}} P_{3n+j} \qquad j = 0, 1, 2.$$

Then $I_0 I_1 I_2$ is zero but

$$I_1 \not\subseteq Q_1, \qquad I_2 \not\subseteq Q_2, \qquad I_3 \not\subseteq Q_3.$$

Hence, $Q_0 \cap Q_1 \cap Q_2$ is not primal. Thus, by the characterisation theorem, Theorem 5.7, $K_s(A) = \frac{1}{2}$ and $K(A) = \frac{1}{\sqrt{3}}$.

The final example is the author's own modification of Example 5.9. See also [32], Example 2.8.

Example 5.10. Let A be the set of sequences (a_n) in $M_2(\mathbb{C})$ such that the subsequences (a_{4r}), (a_{4r+1}), (a_{4r+2}) and (a_{4r+3}) converge in $M_2(\mathbb{C})$ to matrices of the form

$$\begin{pmatrix} \lambda_1(a) & 0 \\ 0 & \lambda_2(a) \end{pmatrix}, \qquad \begin{pmatrix} \lambda_2(a) & 0 \\ 0 & \lambda_3(a) \end{pmatrix},$$

$$\begin{pmatrix} \lambda_3(a) & 0 \\ 0 & \lambda_0(a) \end{pmatrix}, \qquad \begin{pmatrix} \lambda_0(a) & 0 \\ 0 & \lambda_1(a) \end{pmatrix},$$

for some complex numbers $\lambda_0(a)$, $\lambda_1(a)$, $\lambda_2(a)$ and $\lambda_3(a)$. With pointwise defined operations and the supremum norm, A is a C*-algebra. As above, Q_0, Q_1, Q_2 and Q_3 are primitive ideals of A where

$$Q_j = \ker x_j = \{a \in A : \lambda_j(a) = 0\} \qquad j = 0, 1, 2, 3$$

and $Q_1 \cap Q_2$, $Q_2 \cap Q_3$, $Q_3 \cap Q_0$ and $Q_0 \cap Q_1$ are primal and hence contain the same Glimm ideal. Define

$$I_j = \cap_{n \in \mathbb{N}} P_{3n+j} \qquad j = 0, 1, 2, 3.$$

Then $I_0 I_1 I_2 I_3 = \{0\}$ but

$$I_0 \nsubseteq Q_3, Q_0, \qquad I_1 \nsubseteq Q_0, Q_1, \qquad I_2 \nsubseteq Q_1, Q_2, \qquad I_3 \nsubseteq Q_2, Q_3$$

so $Q_0 \cap Q_2$ and $Q_1 \cap Q_3$ are not primal. Hence by the characterisation theorem, Theorem 5.7, $K(A) \geq K_s(A) \geq 1$.

Chapter 6

Related Results

In this chapter we present a brief survey of other results concerning $K(A)$ and $K_s(A)$, and consider how they relate to the theory developed in this paper. Proofs will not always be given.

First, we recall some definitions. A C*-algebra A is said to be **quasicentral** if no primitive ideal of A contains $Z(A)$. Clearly all unital C*-algebras are quasicentral. An equivalence relation is said to be **open** if the quotient map is open. Let X is a subspace of a Banach space Y. Then X is said to be **proximinal** if, for each y in Y, there exists x in X such that

$$\|x - y\| = d(y, X).$$

Let A be a C*-algebra. We define a relation $\sim$ on $\operatorname{Prim} A$ by $P \sim Q$ if and only if P and Q cannot be separated by disjoint open sets. Clearly $\sim$ is reflexive and symmetric, but is not necessarily transitive. By Corollary 2.8, $P \sim Q$ if and only if $P \cap Q$ is primal. It is easy to see that if $P \sim Q$ then $P \approx Q$, but the converse is false. However, [30], Proposition 3.2, states that if A is quasicentral and $\sim$ is an equivalence relation then $\sim$ and $\approx$ agree on $\operatorname{Prim} A$. It is easy to check that for A quasicentral $\sim$ is an equivalence relation if and only if whenever two primitive ideals contain the same Glimm ideal then their intersection is primal. The relations also agree when every Glimm ideal is primal [30].

The class of C*-algebras for which $\sim$ is an open equivalence relation are termed **quasi-standard** [7], Section 1. The quasi-standard C*-algebras include von-Neumann algebras, AW*-algebras, pre-standard algebras, C*-algebras for which the Jacobson topology is Hausdorff and a number of group C*-algebras. They have important connections to the problem of representing a C*-algebra as an algebra of cross sections over a base space. Specifically, a separable C*-algebra is quasi-standard if and only if it is *-isomorphic to a maximal full algebra of cross sections over a base space such that the fibre algebras are primitive throughout a dense subset (see [7] and [30] for details).

The relation $\sim$ also proves important in determining information about $K(A)$ and $K_s(A)$. Let the primitive ideals of a unital C*-algebra A be the nodes of a graph with points P and Q connected if and only if $P \sim Q$. The distance between nodes is the length of the shortest path or infinity if no such path exists. The diameter of a set of nodes is the supremum of the distances between the nodes, with the non-standard convention that the diameter of a singleton set is 1. Then we define $\operatorname{Orc} A$ to be the supremum of the diameters of the connected components of the graph of $\operatorname{Prim} A$ [32], Section 2. The following important theorem is the main result of [32].

Theorem 6.1. *Let A be a non-commutative unital C*-algebra. Then*

$$K_s(A) = \frac{1}{2} \operatorname{Orc} A.$$

Since $\operatorname{Orc} A$ is a positive integer or infinity it follows that $K_s(A)$ is of the form $\frac{n}{2}$, with n a natural number, or infinity. Examples are given in [32], Example 2.8, to show that all of these values occur.

If $\sim$ is an equivalence relation on $\operatorname{Prim} A$ then, by transitivity, the shortest path between two connected points P and Q is $P \sim Q$ and $\operatorname{Orc} A$ is 1. Conversely, if $\operatorname{Orc} A$ is 1 and, for distinct primitive ideals P, Q and R, $P \sim Q$ and $Q \sim R$, then the shortest path between P and R has unit length, and therefore $P \sim R$. Hence, if A is a non-commutative unital C*-algebra, $K_s(A) = \frac{1}{2}$ if and only if $\sim$ is an equivalence relation, if and only if whenever two primitive ideals contain the same Glimm ideal their intersection is primal. Otherwise $\operatorname{Orc} A \geq 2$ so $K_s(A) \geq 1$. Hence we have deduced Theorem 5.3 from Theorem 6.1.

Let A be a von-Neumann algebra. Recall that non-trivial von-Neumann algebras are always unital [23], Theorem 4.1.7. In [36], Zsidó showed that the function $I_x \to \lambda(a_{K_x})$ is continuous on $\operatorname{Prim} Z(A)$. Then by Theorem 2.12, there exists z^a in $Z(A)$ such that $\lambda(a_{K_x}) - z^a$ lies in I_x for all x in $\Delta(Z(A))$. Since each Glimm ideal K_x contains I_x,

$$\lambda(a_G) = z^a_G$$

for all G in $\operatorname{Glimm} A$. Furthermore, Zsidó showed that the norm of $a - z^a$ equalled the distance of a from $Z(A)$, i.e. that $Z(A)$ was proximinal in A. In [31], Corollary 2.5, Somerset extends this result to show that $Z(A)$ is proximinal for any quasi-standard C*-algebra. The question of whether $Z(A)$ is proximinal for a general C*-algebra is open [31].

Using this result and [17], Theorem 4.7, which states that every Glimm ideal of a von-Neumann algebra is primitive, Zsidó argued:

$$\begin{aligned} 2d(a, Z(A)) &= 2\|a - z^a\| \\ &= 2\sup\{\|a_G - \lambda(a_G)\| : G \in \operatorname{Glimm} A\} \\ &\leq 2\sup\{\|a_P - \lambda(a_P)\| : P \in \operatorname{Prim} A\} \\ &= \|ad_A a\|. \end{aligned}$$

Hence $K(A) \leq \frac{1}{2}$ so it was shown that, for a non-commutative von-Neumann algebra, $K(A) = \frac{1}{2}$. Somerset generalised this to the following ([31], Theorem 2.7).

Theorem 6.2. *Let A be a non-commutative unital C*-algebra. If every Glimm ideal of A is primal then $K(A) = \frac{1}{2}$.*

Proof. By Theorem 3.11 and Proposition 4.4,

$$\begin{aligned} 2d(a, Z(A)) &= 2\sup\{\|a_G - \lambda(a_G)\| : G \in \operatorname{Glimm} A\} \\ &\leq 2\sup\{\|a_P - \lambda(a_P)\| : P \in \operatorname{Primal} A\} \\ &= \|ad_A a\|. \end{aligned}$$

Then $K(A) \leq \frac{1}{2}$ and hence $K(A) = \frac{1}{2}$. □

Of course, if every Glimm ideal is primal, every intersection of primitive ideals containing the same Glimm ideal is primal, so we can recover this result from the categorisation theorem, Theorem 5.7.

By [7], Theorem 3.3, equivalence of (i) and (iv), if A is a quasi-standard C*-algebra then $\operatorname{Glimm} A$ is the set of minimal primal ideals. In [31], Lemma 2.8, it is shown that if A is a quotient of a AW*-algebra then each Glimm ideal is prime. Hence we have [31], Corollary 2.9.

Corollary 6.3. *Let A be a non-commutative unital C*-algebra. If A is quasicentral or a quotient of an AW*-algebra then $K(A) = \frac{1}{2}$.*

We now consider a connection between $K(A)$ and the theory of derivations of a C*-algebra A. We begin with some definitions and technical results.

A derivation on a C*-algebra A is said to be a *-derivation if

$$D(a^*) = D(a)^*$$

for all a in A. Let X and Y be normed spaces, and let T be a linear operator, with domain $\mathcal{D}(T)$ in X and range $\mathcal{R}(T)$ in Y. Then T is said to be **closed** if, whenever (x_n) is a sequence in $\mathcal{D}(T)$, convergent to x in X, such that (Tx_n) converges to some y in Y, then x lies in $\mathcal{D}(T)$ and Tx is y. The following is the well known Closed Graph Theorem [22], Theorem 4.13-2.

Theorem 6.4. *Let X and Y be Banach spaces and T a closed linear operator with domain in X and range in Y. If $\mathcal{D}(T)$ is closed in X, then T is bounded.*

The next two results are from the proof of [29], Theorem 2.3.1.

Lemma 6.5. *Let a be a self adjoint element of a unital C*-algebra A. Then there exists a state x such that $|x(a)|$ equals $\|a\|$, and, for any such state, $x(D(a))$ is zero for every *-derivation D on A.*

Proof. Since a is self adjoint, $\|a\|$ lies in its spectrum, and, by Lemma 3.2, such an x exists. Consider the case $x(a)$ equals $\|a\|$. By the functional calculus $\|a\| - a$ is positive. Hence it is equal to b^2 for some positive b. Since $D(1)$ is zero for any derivation,

$$\begin{aligned} |-x(D(a))| &= |x(D(\|a\| - a))| \\ &= \left|x\left(D\left(b^2\right)\right)\right| \\ &\leq |x(bD(b))| + |x(D(b)\,b)|. \end{aligned}$$

By the Schwarz inequality,

$$\begin{aligned} |x(bD(b))|^2 &\leq x(bb^*)\, x\left(D(b)^* D(b)\right) \\ &= x(\|a\| - a)\, x\left(D(b)^2\right) \\ &= 0. \end{aligned}$$

Similarly for $|x(D(b)\,b)|$. Hence $x(D(a))$ is zero. In the case $x(a)$ equals $-\|a\|$, replace a with $-a$. □

Corollary 6.6. *Let A be a unital C*-algebra, D a *-derivative and (a_n) a self adjoint sequence in A such that (a_n) converges to zero and (Da_n) converges to b in A. Then b is zero.*

Proof. Since D is a *-derivative and A_{sa} is norm closed, $(b + a_n)$ is a self adjoint sequence and there exists a sequence (x_n) of states such that, for all n in $\mathbb{N}$,

$$\|b + a_n\| = x_n(b + a_n).$$

Since S_A is weak*-compact, by [21], Chapter 5, Theorem 2, there is a subnet $(x_{n_\lambda})_{\lambda\in\Lambda}$ of (x_n), weak*-convergent to some state x. For all λ in Λ,

$$\begin{aligned} ||x(b)| - \|b\|| &\leq ||x(b)| - |x_{n_\lambda}(b + a_{n_\lambda})|| + ||x_{n_\lambda}(b + a_{n_\lambda})| - \|b\|| \\ &\leq |x(b) - x_{n_\lambda}(b + a_{n_\lambda})| + |\|b + a_{n_\lambda}\| - \|b\|| \\ &\leq |x(b) - x_{n_\lambda}(b)| + |x_{n_\lambda}(a_{n_\lambda})| + \|a_{n_\lambda}\| \\ &\leq |x(b) - x_{n_\lambda}(b)| + 2\|a_{n_\lambda}\|. \end{aligned}$$

Since (x_{n_λ}) weak*-converges to x and (a_{n_λ}) converges to zero it follows that $|x(b)|$ equals $\|b\|$. Now, Lemma 6.5 implies that $x(D(b))$ is zero and that $x_n(D(b + a_n))$ is zero for each n in $\mathbb{N}$. Hence, for all n in $\mathbb{N}$,

$$\begin{aligned} |x(b)| &\leq |x(b - D(a_n))| + |x(D(a_n))| \\ &\leq \|b - D(a_n)\| + |x(D(b + a_n))|. \end{aligned}$$

Given $\varepsilon > 0$, there exists λ in Λ such that

$$\begin{aligned} |x(D(a_n + b))| &= |(x - x_{n_\lambda})(D(b + a_n))| < \tfrac{\varepsilon}{2} \\ \|b - D(a_{n_\lambda})\| &< \tfrac{\varepsilon}{2} \end{aligned}$$

and, hence,

$$|x(b)| = \|b\| = 0.$$

□

We are now able to prove the following important conjecture of Kaplansky. This was proved by Sakai in [28].

Theorem 6.7. *Let A be a unital C*-algebra and let D be a derivation on A. Then D is bounded.*

Proof. Since every sequence can be decomposed into a complex sum of self adjoint sequences, Corollary 6.6 extends by linearity to any sequence convergent to zero. Using linearity again shows that any *-derivation D is a closed linear operator with closed domain A. Hence, by the closed graph theorem, Theorem 6.4, it is bounded. Any derivation D may be decomposed into a complex sum of *-derivatives by

$$D(a) = \frac{D(a^*)^* + D(a)}{2} + i\left(\frac{iD(a^*)^* - iD(a)}{2}\right).$$

Since the *-derivations are bounded it follows that D is bounded. □

It is now easy to check that $\Delta(A)$, the set of all derivations on A, is a closed vector subspace of $B(A)$ and is hence a Banach space.

We now show that the value of $K(A)$ determines whether $\Delta_0(A)$, the set of all inner derivations of A, is norm-closed in $\Delta(A)$. This is implicit in [18], Theorem 5.3. This result is also of historical importance, as it presumably inspired the definition of $K(A)$.

Theorem 6.8. *Let A be a unital C*-algebra. Then $\Delta_0(A)$ is norm closed in $\Delta(A)$ if and only if $K(A)$ is finite.*

Proof. Let Z be the centre of A and let $\psi : A/Z \to \Delta_0(A)$ be the map

$$\psi(a_Z) = \mathrm{ad}_A\, a$$

for a in A. It is easy to check that ψ is a well defined linear bijection. Since

$$\|\psi(a_Z)\| = \|\mathrm{ad}_A\, a\| \leq 2d(a, Z(A)) = 2\|a_Z\|,$$

ψ is bounded, and, since its domain is A/Z, it follows that ψ is a closed linear operator. It then follows from the definition that $\psi^{-1} : \Delta_0(A) \to A/Z$ is a closed linear operator.

Now suppose that $\Delta_0(A)$ is norm closed in $\Delta(A)$. Then, by the closed graph theorem, Theorem 6.4, ψ^{-1} is bounded and

$$d(a, Z(A)) = \left\|\psi^{-1}(\mathrm{ad}_A\, a)\right\| \leq \left\|\psi^{-1}\right\| \|\mathrm{ad}_A\, a\|$$

for all a in A. Hence $K(A)$ is finite.

Conversely, suppose that $K(A)$ is finite. Then, for all a in A,

$$\tfrac{1}{2}\|\mathrm{ad}_A\, a\| \leq \|a_Z\| = d(a, Z(A)) \leq K(A)\|\mathrm{ad}_A\, a\|.$$

Thus, the norm defined on $\Delta_0(A)$ by

$$\|\mathrm{ad}_A\, a\|_1 = \|a_Z\|$$

for a in A is equivalent to the norm induced by $\Delta(A)$. Since A/Z is a Banach algebra it follows that $\Delta_0(A)$ is norm closed in $\Delta(A)$. □

In [8], Batty investigated how properties of the derivations of two C*-algebras relate to the properties of the derivations of their C*-tensor products. Of particular interest to us is the following result. Let $A \otimes_\beta B$ be any C*-tensor product of C*-algebras A and B. Then

$$K(A \otimes_\beta B) \leq 4K(A) + 2K(B) + 4$$

where, without loss of generality, we may interchange A and B to get the smaller bound. Archbold [3] had independently obtained the estimate

$$K(A \otimes_\beta B) \leq 2K(A) + K(B) + 4.$$

Batty also proved [3],

$$K\left(A \otimes_{\beta} B\right) \leq 1+\left(2K\left(A\right)+1\right)\left(2K\left(B\right)+1\right),$$

which gives a smaller bound when $K(A)$ and $K(B)$ are close to $\frac{1}{2}$. The significance of these inequalities with respect to Theorem 6.8 is that they show that if $K(A)$ and $K(B)$ are finite then $K(A \otimes_{\beta} B)$ is finite, i.e. if $\Delta_0(A)$ and $\Delta_0(B)$ are norm closed in $\Delta(A)$ and $\Delta(B)$ respectively then $\Delta_0(A \otimes_{\beta} B)$ is norm closed in $\Delta(A \otimes_{\beta} B)$ for any C*-tensor product $A \otimes_{\beta} B$.

Archbold also investigated the behaviour of $K(A)$ under ideals and quotients in [3]. He gave an example of a C*-algebra with ideal J such that $K(A)$ was $\frac{1}{2}$ but $K(A/J)$ was infinite. In fact, as was observed in [30], $K(A)$ is very unstable under quotients. Every unital C*-algebra A may be considered as a quotient of a unital quasi-standard C*-algebra B [30], Proposition 3.7, but then $K(B)$ is $\frac{1}{2}$ regardless of the value of $K(A)$. However Archbold [3], Proposition 3, showed that for ideals, the following holds:

Proposition 6.9. *If A is a C*-algebra and J is an ideal*

$$\|\mathrm{ad}_{A} a\| = \|\mathrm{ad}_{J} a\|$$

for all a in J. It then follows that

$$K(J) \leq 2K(A).$$

Since we have concentrated on unital C*-algebras in this paper, it is useful to know that Proposition 6.9 implies that, for a non-unital C*-algebra A,

$$\tfrac{1}{2}K(A) \leq K(A+\mathbb{C}1) \leq K(A).$$

Somerset improves on this result when J is quasicentral in [30], Proposition 5.3, where he shows that

$$d(a, Z(J)) = d(a, Z(A))$$

for all a in J. Combining this with Proposition 6.9 shows that

$$d(a, Z(J)) = d(a, Z(A)) \leq K(A)\|\mathrm{ad}_A a\| = K(A)\|\mathrm{ad}_J a\|$$

for all a in J, from which it follows that $K(J) \leq K(A)$. Hence, if A is a non-unital quasicentral C*-algebra,

$$K(A) \leq K(A+\mathbb{C}1) \leq K(A),$$

so the problem of computing $K(A)$ is reduced to the unital case.

A C*-algebra is **weakly central** if for maximal ideals M and N of A

$$M \cap Z(A) = N \cap Z(A) \Leftrightarrow M = N.$$

Archbold proves the following result in [3], Theorem 4.1, using Katetov's interpolation theorem.

Theorem 6.10. *Let A be a weakly central unital C*-algebra and let J be an ideal of A. Then for self adjoint a in J there exists self adjoint z in $Z \cap J$ such that*

$$\|a-z\| \leq \|\mathrm{ad}_A a\|.$$

In particular $K_s(A) \leq 1$.

Somerset [30], Theorem 4.1, uses Helly's Theorem to show that, when A is weakly central, $K(A)$, and hence $K_s(A)$, are bounded above by 1.

Example 5.9 shows that $K(A)$ and $K_s(A)$ may have different values. As shown in Chapter 1,

$$\frac{K(A)}{K_s(A)} \leq 2.$$

Somerset [30] conjectures that

$$\frac{K(A)}{K_s(A)} \leq \frac{2}{\sqrt{3}},$$

the value achieved in Example 5.9. He shows in [30], Proposition 5.15, that

$$\frac{K_n(A)}{K_s(A)} \leq \frac{2}{\sqrt{3}},$$

where $K_n(A)$ is the least element of $[0, \infty]$ such that

$$d(a, Z(A)) \leq K_n(A) \|\mathrm{ad}_A a\|$$

for all normal a in A. He further observes that there is no known example for which $K_n(A)$ and $K(A)$ disagree. This might suggest the conjecture that $K_n(A)$ and $K(A)$ are always equal. However, the lack of a counter example is more likely to be due to the lack of tools for calculating $K_n(A)$ and $K(A)$ in general. When A is a unital C*-algebra and $K(A)$ is bounded above by 1 we can make the following remark (apparently not stated elsewhere).

Remark 6.11. *Let A be a unital C*-algebra with $K(A)$ bounded above by 1. Then $K(A)$ and $K_n(A)$ are equal.*

Proof. By the categorisation theorem, Theorem 5.7, $K(A)$ and $K_s(A)$ agree if $K(A)$ is 1 or $\frac{1}{2}$. This forces $K_n(A)$ to agree with $K(A)$ and $K_s(A)$. Re-examining the proof of Theorem 5.5 we see that we in fact proved that when A is a unital C*-algebra with three primitive ideals containing the same Glimm ideal which have non-primal intersection then $K_n(A)$ is bounded below by $1/\sqrt{3}$. Since $K(A)$ is $1/\sqrt{3}$ in this case we have

$$K(A) = K_n(A) = \tfrac{1}{\sqrt{3}}.$$

□

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