Factorial Functionals

and

Primal Ideals

of

JB*-triples

Christopher Hoskin

Mathematical Institute

University of Oxford

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Abstract

The research presented in this thesis furthers the ongoing investigation into the structure of JB*-triples, an important class of Banach space with applications to many areas of mathematics and mathematical physics. The thesis initiates the study of the connected theories of factorial functionals and primal ideals in the general JB*-triple situation and then gives applications of these theories, including:

(i) a non-abelian analogue of the Gelfand representation over a base space of minimal primal ideals;

(ii) an investigation into the primitivity of minimal primal ideals;

(iii) a characterisation of prime JB*-triples in terms of finite factorial functionals;

(iv) a necessary condition on the factorial functionals for a JB*-triple to be antiliminal;

(v) a characterisation of elements in the pure functional space of a continuous JBW*-triple.

Application (i) provides a tool for studying the structure of a class of JB*-triples. In particular it applies to JBW*-triples. Applications (i) and (ii) lead to a Gelfand representation of Type I JBW*-triples with primitive fibres. Applications (iii) and (iv) are connected to Stone-Weierstrass theorems for JB*-triples. Application (v) is of interest because of the theoretical importance of pure functionals, and because pure functionals represent the pure states in quantum mechanical models.

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Contents

Chapter 1

Introduction

The study of JB*-triples has its origins in the study of bounded symmetric domains. The algebraic theory was initiated by Koecher [60], Meyberg [63] and Loos [62]. In particular Koecher was able to obtain Cartan's classification of bounded symmetric domains in $\mathbb{C}^n$ using Jordan algebras. This approach was more readily adapted to infinite dimensional complex Banach spaces than Cartan's Lie algebra method. It transpired from the work of Kaup, Upmeier and Vigué [58], [72] [74], [75] that the holomorphic structure of the open unit ball of a complex Banach space A induces a triple product $\{\ldots\}$ from $A \times A_s \times A$ to A, where A_s is a closed subspace of A called the symmetric part. In extreme cases, A_s may be zero, but the Banach spaces A such that A coincides with A_s are the JB*-triples. Kaup proved conclusively that JB*-triples are a natural class of objects by showing that the category of JB*-triples is equivalent to the category of bounded symmetric domains with base point by means of the mapping taking the JB*-triple A to its open unit ball.

JB*-triples were subsequently found to have applications to other areas of mathematics and mathematical physics; some of these are explored in the exposition [71]. From a more abstract point of view, the class of JB*-triples is of interest as it includes many important subclasses, such as C*-algebras, JB*-algebras, Hilbert C*-modules and spin triples. In axiomatic models of statistical physical systems (e.g. quantum mechanics) the conditional expectations of classical probability theory are replaced by contractive projections. However, neither the class of C*-algebras or the larger class of JB*-algebras usually used in such models are closed under all contractive projections. The fact that the category of JB*-triples is closed under all contractive projections, and thus suitable for use in these axiomatic models, is perhaps the most persuasive argument for its study [59].

These facts have inspired an extensive study into the structure of JB*-triples by several authors (see the References section). It has been shown that many well known results from the theory of C*-algebras are in fact consequences of JB*-triple results. The lack of a global order structure or an underlying Hilbert space in the JB*-triple case often necessitates new or modified proofs, which in turn leads to new insights and even new results for C*-algebras. This will be demonstrated in this thesis, as existing results in the theory of factorial states and primal ideals of C*-algebras may be recovered from our results.

This thesis has two primary aims; to investigate a Gelfand-Naimark representation for JB*-triples (Chapters 3-6) and to study the factorial functionals of JB*-triples (Chapter 7). The connection between these two topics is the involvement of primal ideals in both theories.

To better describe the results obtained, it will be convenient to first introduce some concepts and notation.

1.1 Concepts and notation

Let A be a JB*-triple and let $Z(A)$ be the set of bounded operators T from A to itself, satisfying, for elements a, b and c in A,

$$T\{a\,b\,c\} = \{T a\,b\,c\}.$$

The space $Z(A)$ is said to be the *centralizer* of A and is a commutative unital C*-algebra with involution $T \mapsto T^\dagger$ satisfying

$$T\{a\,b\,c\} = \{a\,T^\dagger b\,c\}$$

for all elements T in $Z(A)$ and a, b and c in A.

A JB*-triple A possessing a (necessarily unique) predual A_* is said to be a *JBW*-triple*. The bidual A^{**} of a JB*-triple A is always a JBW*-triple. A subspace J of the JB*-triple A is said to be an *ideal* in A, when $\{J\,A\,A\}$ and $\{A\,J\,A\}$ are contained in J. This is exactly what is required for the triple product on A to induce a triple product on the quotient space A/J, and when J is closed, J and A/J are themselves JB*-triples. When A is a JBW*-triple and J is a w*-closed ideal of A, A/J has a natural identification with a w*-closed ideal $J^{\perp}$ of A, and A can be decomposed into an l^{∞} sum of J and $J^{\perp}$. A JBW*-triple A is said to be a *factor* if the only such decomposition is the trivial one. The factors can be given a Murray von-Neumann type classification. Thus factors are the basic building blocks from which more complex JBW*-triples are built.

The discrete factors in the Murray von-Neumann classification are known as *Cartan factors*, and reduce to the classical Cartan factors in the finite-dimensional situation. The *pure functionals* of a JB*-triple A are the elements of the set $\partial_e A_1^*$ of extreme points of the unit ball of A^*, the dual of A. Each element x of A^* has a w*-closed ideal M_x of A^{**} associated with it. The w*-closed ideals M_x for x in $\partial_e A_1^*$ are exactly the Cartan factors contained in A^{**}. There exists a natural triple homomorphism π_x from A onto a w*-dense subset of M_x. An ideal of A is said to be *primitive* if it is the kernel $\ker \pi_x$ of π_x for some x in $\partial_e A_1^*$. The set of primitive ideals is denoted by $\operatorname{Prim} A$. A JB*-triple is said to be *primitive* if the zero ideal is primitive. In particular, for every x in $\partial_e A_1^*$, $A/\ker \pi_x$ is primitive and π_x induces an injective homomorphism identifying $A/\ker \pi_x$ with a w*-dense subtriple of M_x. Cartan factors and primitive ideals play a fundamental role in the theory. For example, the main result of [45] is a Gelfand-Naimark theorem for JB*-triples.

The classes of factorial functionals and primal ideals referred to in the title of this thesis are enlargements of the classes of pure functionals and primitive ideals respectively. A functional x in A^*, the dual of a JB*-triple A, is said to be *factorial* if the w*-closed ideal M_x of the bidual A^{**} associated with x is a factor. A norm-closed ideal J of the JB*-triple A is said to be *primal* if whenever $J_1, \ldots, J_n$ are norm-closed ideals of A such that $\cap_{j=1}^{n} J_j$ is zero, J_j is a subset of J for some j in $1, \ldots, n$.

A more detailed summary of the theory of JB*-triples is presented in Chapter 2.

1.2 A Gelfand-Naimark representation

The first aim of this thesis is to produce a Gelfand representation for JB*-triples. Given a JB*-triple A, the object is to find a locally compact Hausdorff space, Ω, known as the *base space*, associate to each point ω in Ω a JB*-triple A_ω, the *fibre* at ω, and represent each element a of A as an element $\hat{a}$ of $\prod_{\omega \in \Omega} A_\omega$. For this representation to be of use in the study of A, the representation must decompose the structure of A into the structure of the fibres, and the fibres must belong to an accessible subclass of the class of JB*-triples. To this end, a representation satisfying the following conditions is sought:

(S1) the range of the representation is a structure known as a maximal full triple of cross-sections (defined in Chapter 5);

(S2) the fibres on a dense subset of the base-space are primitive.

A JB*-triple possessing a representation satisfying conditions S1 and S2 is said to be *densely standard*. The motivation for and utility of constructing a Gelfand representation of a densely standard JB*-triple can be seen by considering the example of an associative JB*-algebra A. Take Ω to be $\operatorname{Prim} A$, the primitive spectrum of A, and let $\Omega(A)$ denote the character space equipped with the w*-topology. Then, the map $x \mapsto \ker x$ is a homeomorphism from $\Omega(A)$ onto $\operatorname{Prim} A$, and $\operatorname{Prim} A$ coincides with the set of maximal modular ideals of A. For each element x in $\Omega(A)$, the primitive quotient $A/\ker x$ can be canonically identified with the complex field, and, for all elements a in A,

$$\hat{a}(\ker x) = x(a).$$

Thus, the map $\ker x \mapsto a + \ker x$ is the usual Gelfand representation $x \mapsto x(a)$ of a. The centralizer of $C_0(\Omega)$, the continuous functions on Ω vanishing at infinity, is *-isomorphic to $C^b(\Omega)$, the continuous bounded functions on Ω ([64], Example 3.1.3).

It is shown (Proposition 5.9, Theorem 5.10, Theorem 5.13) that if A is a JB*-triple possessing a representation satisfying condition S1 then the representation decomposes pure functionals and norm-closed ideals of A into the corresponding objects in the fibres, and a Glimm Stone-Weierstrass type theorem holds. Condition S1 also allows Ω to be identified with a set of norm-closed ideals of A, constructed from the primitive spectrum of A (Theorem 5.16). To investigate the implications of imposing condition S2 on the representation, the weaker condition

(S2$'$) Ω possesses a dense subset of primal ideals

is introduced. A JB*-triple is said to be *quasi-standard* if it possesses a representation satisfying conditions S1 and S2$'$. It is shown (Theorem 5.17) that, for a quasi-standard JB*-triple, Ω is unique up to homeomorphism and may be identified with the complete regularisation of Prim A, and with the set of minimal primal ideals. Furthermore, the representation transforms the action of the centralizer $Z(A)$ of A into the action of pointwise multiplication by elements of $C^b(\Omega)$, the space of continuous bounded complex-valued functions on Ω (Corollary 5.18).

The uniqueness of the base space of a quasi-standard JB*-triple simplifies the process of determining which JB*-triples are quasi-standard and which are densely standard. It is shown (Theorem 6.13) that all JBW*-triples are quasi-standard and that Type I JBW*-triples are densely standard (Theorem 6.24). The class of JB*-triples with Hausdorff primitive spectrum is densely standard (Theorem 5.21). This class includes some interesting subclasses such as JB*-triples of finite rank and abelian JB*-triples, even though abelian JB*-triples need not be associative JB*-algebras. There are examples of JB*-triples, even in the class of C*-algebras, which are not quasi-standard ([73], p296).

1.3 Factorial functionals

Factorial functionals have been studied in arbitrary Banach spaces ([28]) and in ordered spaces ([27], [78]) where attention is restricted to those factorial functionals which are also states. Factorial states have been studied in connection with the central decomposition theorem for JB*-algebras ([78]) and with various Stone-Weierstrass problems in the C*-algebra case ([5], [16]).

This thesis begins the development of the theory of factorial functionals for an arbitrary JB*-triple. It is shown that a JB*-triple is prime if and only if the factorial functionals are w*-dense in the surface of the dual unit ball (Proposition 7.21) and that the set of pure functionals of an antiliminal JB*-triple are w*-dense in the set of factorial functionals. Together these results form a partial decomposition of [21], Theorem 5.6, a key step in the proof of the important Glimm Stone-Weierstrass theorem for JB*-triples ([67], Theorem 6.2). In the process, a number of interesting technical results are obtained, including the construction of Type I factorial functionals as σ-convex sums of pure functionals (Proposition 7.10) and a characterisation of primal ideals of JB*-algebras in terms of w*-density of factorial functionals in faces of the state space (Theorem 7.11).

It is shown that every functional in the dual unit ball of a JB*-triple induces a state on the centralizer. The theory of factorial functionals is then used to deduce the final theorem of the thesis, a characterisation of the pure functional space of a continuous JBW*-triple in terms of those functionals which induce characters of the centralizer (Theorem 7.31).

1.4 Structure of material

In Chapter 2, the basic theory of JB*-triples is summarised.

In Chapter 3, the lattice of M-ideals of a Banach space is studied. Many of the results depend on purely lattice theoretic considerations, and will be introduced in this context to aid clarity.

In Chapter 4, some results concerning central kernels are established for later use and the discussion of Chapter 3 is specialised to the ideal structure of JB*-triples. A number of facts connected with the primitive spectrum of a JB*-triple are established.

In Chapter 5, the results of Chapter 3 and Chapter 4 are applied to construct representations of JB*-triples over base spaces of primal ideals.

In Chapter 6, the results of Chapter 5 are applied to show that JBW*-triples are quasi-standard and, furthermore, that Type I JBW*-triples are densely standard.

In Chapter 7, the factorial functionals of a JB*-triple and their connection to primal ideals are studied. A number of applications to the structure theory of JB*-triples are given.

Chapter 2

JB*-triples

JB*-triples are the principal object of study in this thesis. JB*-triples were originally introduced by Kaup [58] in the context of infinite dimensional holomorphy, but their suggested role in quantum mechanics has led to an extensive investigation of their algebraic and geometric properties. For the purposes of this thesis, it will be convenient to consider a JB*-triple as a vector space equipped with an algebraic structure (a Jordan *-triple product) and a Banach norm in such a way that the algebraic structure is intimately intertwined with the geometry of the Banach space. This will be seen to be analogous to the usual view of C*-algebras as *-algebras equipped with a Banach algebra norm satisfying the C*-condition. Indeed, C*-algebras are an important sub-class of the JB*-triples.

The aforementioned algebraic structures, Jordan *-triples, are less well known than *-algebras and it is therefore necessary to review briefly their properties in Section 2.1 before introducing JB*-triples in Section 2.2. In the process, a sub-class of the JB*-triples will be encountered, the JB*-algebras. The underlying algebraic structure of a JB*-algebra is a commutative non-associative *-algebra known as a Jordan *-algebra. JB*-algebras play an important part in the theory of JB*-triples, as well as being objects of great interest in their own right. The class of JB*-algebras includes the class of C*-algebras.

The class of JBW*-triples has a more complete and satisfactory theory than the class of JB*-triples. The bidual of every JB*-triple is a JBW*-triple. Therefore the

JBW*-triples play an important role in the theory.

2.1 Algebraic Jordan structures

2.1.1 Jordan algebras

This section introduces Jordan algebras, following [51], Chapter 2, to which the reader is referred for further details. Jordan *-algebras are important both as a subclass of the Jordan *-triples introduced in Section 2.1.2, and as a tool in the development of their theory.

An *algebra* A is a real or complex vector space with a bilinear multiplication $(a, b) \mapsto ab$ from $A \times A$ to A. We define operators L_a, R_a from A to A by

$$L_a b = ab \qquad R_a b = ba$$

for all a and b in A. A complex algebra equipped with an involution is said to be a **-algebra.*

Powers of an element a of A are defined inductively by

$$a^1 = a; \qquad a^n = aa^{n-1}; \qquad n \geq 2.$$

The algebra A is said to be *commutative* if ab equals ba and *associative* if $(ab)c$ equals $a(bc)$ for all a, b and c in A. If A is a commutative algebra with multiplication $(a, b) \mapsto a \circ b$ satisfying the Jordan axiom,

$$a \circ (b \circ a^2) = (a \circ b) \circ a^2, \qquad a, b \in A,$$

then A is said to be a *Jordan algebra.* In a Jordan algebra A, the power law

$$a^{n+m} = a^n \circ a^m \qquad n, m \in \mathbb{N}$$

holds for all elements a in A. A Jordan multiplication $(a, b) \mapsto a \circ b$ can be defined on any associative algebra A by

$$a \circ b = \frac{1}{2}(ab + ba), \qquad a, b \in A.$$

When a Jordan algebra is of this form it is said to be *special* and the product $\circ$ is said to be the special product. Otherwise it is said to be *exceptional*. The powers of a special Jordan algebra agree with the powers in the underlying associative algebra.

Let A be a Jordan algebra. Elements a and b of A are said to *operator commute* if the operators L_a and L_b commute. An element of A which operator commutes with all other elements of A is said to be *central*. The *centre* $Z(A)$ of A is defined to be the associative subalgebra of central elements of A. A *Jordan *-algebra* is a Jordan algebra equipped with an involution $*$ such that, for all elements a and b in A,

$$(a \circ b)^* = b^* \circ a^* = a^* \circ b^*.$$

A self-adjoint idempotent of a Jordan *-algebra A is said to be a *projection*. The set of projections is denoted by $\mathcal{P}(A)$ and the set of central projections by $\mathcal{ZP}(A)$. Projections p and q of A are said to be *orthogonal*, written $p \perp q$ if $p \circ q$, is zero.

2.1.2 Jordan *-triples

A **-triple* over $\mathbb{C}$ is a complex vector space A with triple product $\{\ldots\} : A^3 \mapsto A$ which is symmetric and linear in the first and third variables and conjugate linear in the second. For any elements a and b of A, operators $D(a, b) : A \mapsto A$, $Q(a, b) : A \mapsto A$ and $\circ_a : A \times A \mapsto A$ are defined by

$$D(a, b)t = \{a\, b\, t\} \qquad Q(a, b)(t) = \{a\, t\, b\} \qquad r \circ_a s = \{r\, a\, s\}$$

for all elements r, s and t of A. We write $D(a)$ for $D(a, a)$ and $Q(a)$ for $Q(a, a)$.

A *Jordan *-triple* is a *-triple A satisfying the following weak associativity for all

a, b, r, s in A:

$$\{r\,a\,\{r\,b\,r\}\} = \{r\,\{a\,r\,b\}\,r\}$$
$$\{\{r\,a\,r\}\,a\,s\} = \{r\,\{a\,r\,a\}\,s\}$$

Example 2.1 *Let A be a Jordan *-algebra. Then A is a Jordan *-triple with respect to the triple product defined for a, b and c in A by*

$$\{a\,b\,c\} = (a \circ b^*) \circ c - (a \circ c) \circ b^* + a \circ (b^* \circ c).$$

*When A is an associative *-algebra with the special product,*

$$\{a\,b\,c\} = \frac{1}{2}(ab^*c + cb^*a).$$

A number of 'triple identities' follow from the definition of a Jordan *-triple by linearisation. See [62] for a comprehensive list and proofs.

Theorem 2.2 *A *-triple A is a Jordan *-triple if and only if it satisfies the following identity for all elements a, b, r, s and t of A:*

$$\{a\,b\,\{r\,s\,t\}\} = \{\{a\,b\,r\}\,s\,t\} - \{r\,\{b\,a\,s\}\,t\} + \{r\,s\,\{a\,b\,t\}\}.$$

For any element a of the Jordan *-triple A, the pair $(A, \circ_a)$ is a commutative algebra known as the *a-homotope*. Using the triple identities, it can be seen that $(A, \circ_a)$ is a Jordan algebra.

We define odd powers of an element a of A by

$$a^1 = a; \qquad a^{2n-1} = \{a\,a^{2n-3}\,a\}; \qquad n \geq 2.$$

An inductive argument shows that for all a in A and odd powers m_1, m_2 and m_3, the power law

$$\{a^{m_1}\,a^{m_2}\,a^{m_3}\} = a^{m_1+m_2+m_3}$$

is satisfied. The Jordan *-triple A is said to be *abelian* if, for all a, b, c and d in A, the operators $D(a,b)$ and $D(c,d)$ commute.

A subspace J of A is said to be a *subtriple* if $\{J\,J\,J\}$ lies in J, an *ideal* if $\{A\,A\,J\}+\{A\,J\,A\}$ lies in J and an *inner ideal* if $\{J\,A\,J\}$ lies in J. Clearly ideals are inner ideals and inner ideals are subtriples. When J is an ideal, the quotient space A/J is a Jordan *-triple in the natural way.

An element e in A is said to be a *tripotent* if $\{e\,e\,e\}$ equals e. The set of tripotents is denoted by $\mathcal{U}(A)$. The proof of Theorem 2.3 may be found in [62], Theorem 5.4.

Theorem 2.3 *Let A be a Jordan *-triple and let e be a tripotent in A. Let $D(e)$ be the linear operator and let $Q(e)$ be the conjugate linear operator defined for a in A by*

$$D(e)a = \{e\,e\,a\}, \qquad Q(e)a = \{e\,a\,e\},$$

let $P_0(e)$, $P_1(e)$ and $P_2(e)$ be the linear operators defined by

$$P_0(e) = I - 2D(e) + Q(e)^2, \quad P_1(e) = 2(D(e) - Q(e)^2), \quad P_2(e) = Q(e)^2,$$

and let

$$A_j(e) = \begin{cases} P_j(e)A & j \in \{0,1,2\} \\ \{0\} & j \in \mathbb{Z}\backslash\{0,1,2\} \end{cases}$$

Then, the following results hold.

(i) *The operators $D(e)$ and $Q(e)$ satisfy the equations*

$$Q(e) = Q(e)^3 = D(e)Q(e) = Q(e)D(e).$$

(ii) *The operators $P_0(e)$, $P_1(e)$ and $P_2(e)$ are mutually orthogonal projections, sat-*

isfying the equations

$$\begin{aligned} P_0(e) &= (I - D(e))(I - 2D(e)) \\ P_1(e) &= 4D(e)(I - D(e)) \\ P_2(e) &= D(e)(2D(e) - I) \end{aligned}$$

and

$$\mathrm{Id}_A = P_0(e) + P_1(e) + P_2(e), \qquad D(e) = \tfrac{1}{2}(0P_0(e) + 1P_1(e) + 2P_2(e)).$$

(iii) *A has the decomposition*

$$A = A_0(e) \oplus A_1(e) \oplus A_2(e)$$

(iv) *For j equal to $0, 1, 2$, $A_j(e)$ is the $\frac{j}{2}$-eigenspace of $D(e)$.*

(v) *For j, k and l in $\{0, 1, 2\}$, the multiplication rules*

$$\begin{aligned} \{A_j(e)\, A_k(e)\, A_l(e)\} &\subseteq A_{j-k+l}(e) \\ \{A_2(e)\, A_0(e)\, A\} = \{A_0(e)\, A_2(e)\, A\} &= 0 \end{aligned}$$

are satisfied.

(vi) *The spaces $A_0(e)$ and $A_2(e)$ are inner ideals of A, whilst $A_1(e)$ is a subtriple of A.*

(vii) *The inner ideal $A_2(e)$ is a Jordan *-algebra with unit e under multiplication $(a, b) \mapsto a \circ_e b$ and involution $a \mapsto a^{*_e}$ defined by $a \mapsto Q(e)a$, and the triple product arising from this Jordan *-algebra coincides with the restriction of the triple product of A to $A_2(e)$.*

In the situation described in Theorem 2.3, for j equal to $1, 2, 3$, $P_j(e)$ is known as the *Peirce-j projection* and $A_j(e)$ is the *Peirce-j space* associated with the tripotent

e. The multiplication rules specified in Theorem 2.3 are known as the *Peirce rules*.

A tripotent e is said to be *abelian* if the Peirce 2-space $A_2(e)$ is an abelian JB*-triple, *minimal* if $A_2(e)$ is just the space of scalar multiples of e and *complete* if the Peirce-0 space $A_0(e)$ is zero.

Lemma 2.4 *Let A be a Jordan *-triple and let u and v be tripotents in A. Then the following conditions are equivalent:*

$$\begin{aligned} D(u)v = 0; &\quad D(v)u = 0; \\ v \in A_0(u); &\quad u \in A_0(v); \\ D(u,v) = 0; &\quad D(v,u) = 0. \end{aligned}$$

When the conditions of Lemma 2.4 hold, u and v are said to be *orthogonal* [33].

Proposition 2.5 *Let A be a Jordan *-triple and let $\mathcal{U}(A)$ be the set of tripotents of A. Let $\leq$ be the relation defined on $\mathcal{U}(A)$ by $u \leq v$ if and only if $v - u$ is a tripotent and $v - u$ is orthogonal to u. Let u and v be tripotents of A such that $u \leq v$ and let $A_2(u)$ and $A_2(v)$ be the Peirce-2 spaces of u and v respectively. Then the following results hold.*

(i) *An element w of A is a tripotent $w \leq v$ if and only if w is a projection in $A_2(v)$.*

(ii) *$A_2(u)$ is an inner ideal of the inner ideal $A_2(v)$.*

(iii) *The Jordan *-algebra $A_2(u)$ is a *-subalgebra of the Jordan *-algebra $A_2(v)$.*

(iv) *The relation $\leq$ is a partial ordering of $\mathcal{U}(A)$.*

2.1.3 Anisotropic Jordan*-triples

A Jordan*-triple A is said to be *anisotropic* if a in A is zero if and only if a^3 is zero. Let A be an anisotropic Jordan*-triple. Then for a and b in A, $D(a,b)$ is zero if and only if $D(b,a)$ is zero. Elements a, b satisfying these equivalent conditions are said to be *orthogonal*, written $a \perp b$. Note that when a and b are tripotents, this definition coincides with the definition given in Lemma 2.4. The *algebraic annihilator* of a subset B of A is the set of elements orthogonal to all elements of B, denoted by $B^\perp$.

Lemma 2.6 *Let A be an anisotropic Jordan *-triple and let B and C be subsets of A. Then:*

(i) *$B^\perp$ is an inner ideal of A;*

(ii) *$B \cap B^\perp = \{0\}$;*

(iii) *$B \subseteq B^{\perp\perp}$;*

(iv) *if $B \subseteq C$ then $C^\perp \subseteq B^\perp$;*

(v) *if B is a subtriple of A, then $B \oplus B^\perp$ is a subtriple of A, containing B and $B^\perp$ as ideals.*

The *kernel* $\operatorname{Ker} B$ of a non-empty subset B of A is defined by

$$\operatorname{Ker} B = \{a \in A : \{B\, a\, B\} = 0\}.$$

Clearly $B^\perp$ is contained in $\operatorname{Ker} B$ and $B \cap \operatorname{Ker} B$ is contained in $\{0\}$. A subtriple B of A is said to be *complemented* if

$$A = B \oplus \operatorname{Ker} B.$$

A linear projection P on A is said to be a *structural projection* if

$$PQ(a)P = Q(Pa)$$

for all a in A.

Proposition 2.7 *Let A be an anisotropic Jordan *-triple. Then:*

(i) *a complemented sub-triple B of A is an inner ideal in A such that*

$$\{A\, B\, \operatorname{Ker} B\} \subseteq \operatorname{Ker} B;$$

(ii) *the range of a structural projection P is a complemented sub-triple with kernel* $\mathrm{Ker}(\mathrm{range}\, P)$*;*

(iii) *if B is a complemented sub-triple of A, the linear projection P with range B and kernel* $\mathrm{Ker}\, B$ *is a structural projection on A.*

Lemma 2.8 gives an important example of a complemented subspace.

Lemma 2.8 *Let A be an anisotropic Jordan *-triple and let u be a tripotent in A. Then*

$$A_2(u)^{\perp} = A_0(u)$$

and $A_2(u)$ and $A_0(u)$ are complemented, with

$$\mathrm{Ker}\, A_2(u) = A_1(u) \oplus A_0(u), \qquad \mathrm{Ker}\, A_0(u) = A_2(u) \oplus A_0(u).$$

The results of this section can be found in [34].

2.2 JB*-triples

In this section the situation in which the algebraic structures introduced in Section 2.1 possess a complete norm is considered. Let A be a Banach space and let S be a subset of A. In the sequel, $B(A)$ will be understood to refer to the Banach space of bounded linear operators on A and $\overline{S}^n$ will be understood to refer to the closure of S in the topology induced on A by the norm. When A has a distinguished predual A_*, $\overline{S}^{w*}$ will denote the closure of S in the w*-topology induced on A by A_*.

2.2.1 JB*-algebras

A Jordan *-algebra A equipped with a norm with respect to which it is a Banach algebra is said to be a *JB*-algebra* if all elements a in A satisfy

$$\|a\| = \|a^*\|, \qquad \|\{a\,a\,a\}\| = \|a\|^3.$$

A C*-algebra is a JB*-algebra with respect to the multiplication given by the special product.

A real Jordan algebra A equipped with a norm with respect to which it is a Banach algebra is said to be a *JB-algebra* if all elements a and b in A satisfy

$$\|a^2\| = \|a\|^2, \qquad \|a^2\| \leq \|a^2 + b^2\|.$$

The self-adjoint part of any JB*-algebra is a JB-algebra. A deeper result is that every JB-algebra is the self adjoint part of a unique JB*-algebra [79], Theorem 2.8. The set of squares in a JB-algebra form a proper convex closed cone and a JB*-algebra will always be assumed to have the partial order induced by this cone ([51], Lemma 3.3.7). An exposition of the theory of JB-algebras is given in [51], and we use this to deduce some facts about JB*-algebras.

A JB*-algebra possessing a predual A_* is said to be a *JBW*-algebra*. Let A_*^+ be the cone of positive functionals in A_*. The set

$$S_*(A) = \{x \in A_*^+ : \|x\| = 1\}$$

is said to be the *normal state space* of A. When A is a JBW*-algebra, $\mathcal{P}(A)$ is a complete orthomodular lattice ([51], Lemma 4.2.8). The second dual A^{**} of a JB*-algebra A is a JBW*-algebra which has a separately w*-continuous multiplication and contains A as a sub-algebra ([51], Theorem 4.4.3). The *state space* of A, $S(A)$ is defined to be the normal state space of A^{**}.

The convex geometry of the normal state space of a JBW*-algebra plays a significant role in the theory of JBW*-algebras. Recall that a convex subset F of a convex set C is said to be a *face* of C if, for all x and y in C and λ in $(0, 1)$ such that $\lambda x + (1 - \lambda)y$ lies in F, x and y lie in F. A singleton face is said to be an *extreme point* of C. Let K be a convex subset of a vector space V such that the hyperplane containing K does not contain the origin. Then a face F of K is said to be a *split face* if there exists a face $F^\perp$ of K such that for every x in K there exists unique t in

$[0, 1]$, y in F and z in $F^{\perp}$ such that x can be written as

$$x = ty + (1 - t)z$$

and y and z are unique when t lies in $(0, 1)$. Let V be a real vector space and let V^+ be a proper convex cone in V. An element x of V is said to be an *order unit* for V if, for all y in V, there exists λ in $\mathbb{R}^+$ such that

$$-\lambda x \leq y \leq \lambda x.$$

A real vector space V with order unit x is said to be an *order unit space* if, for y in V, $ny \leq x$ for all n in $\mathbb{N}$ if and only if y equals 0. The *order norm* of the order unit space V is the norm defined for y in V by

$$\|y\| = \inf\{\lambda > 0 : -\lambda x \leq y \leq \lambda x\}.$$

A functional x on a JB*-algebra A is said to be *faithful* if, for a in A, $x(a^* \circ a)$ equals zero implies a equals zero.

Theorem 2.9 *Let A be a JBW*-algebra with predual A_*. Let $A_{*,sa}$ be the self-adjoint part of A_* and let A_*^+ be the cone of positive elements of A_*. Let x be an element of $S_*(A)$, the normal state space of A. Let $\mathrm{face}\, x$ be the face of $S_*(A)$ generated by x, let V_x^+ be the face of A_*^+ generated by x, and let V_x be the subspace of $A_{*,sa}$ generated by V_x^+. Then:*

(i) $V_x^+ = \{y \in A_*^+ : 0 \leq y \leq \lambda x \text{ for some } \lambda \in \mathbb{R}^+\}$;

(ii) *x is an order unit for V_x;*

(iii) $V_x = V_x^+ - V_x^+ = \{y \in A_* : -\lambda x \leq y \leq \lambda x \text{ for some } \lambda \in \mathbb{R}^+\}$;

(iv) *the following are equivalent:*

(a) $\overline{\mathrm{face}\, x}^n = S_*(A)$;

(b) $\overline{V_x^+}^n = A_*^+$;

(c) $\overline{V_x}^n = A_{*,sa}$;

(d) x is faithful.

Proof. Statements (i) to (iii) are easily verified. Observe that $A_{*,sa}$ is the predual of the JB-algebra A_{sa} of self-adjoint elements of A. Equivalence of (a) and (b) is elementary, as is (b)⇒(c) and (c)⇒(d). The result (d)⇒(b) is due to King (see [55], Appendix 2, Lemma 9). ■

The following result may be found in [50].

Proposition 2.10 *Let A be a JBW*-algebra with centre $Z(A)$, let $A_{*,1}$ be the unit ball in the predual and let $S_*(A)$ be the normal state space of A. For a projection p in $Z(A)$, define*

$$\{p\}_{\prime} = \{x \in A_{*,1} : x(p) = 1\}.$$

Then $\{p\}_{\prime}$ is a norm-closed face of $S_*(A)$ and the map $p \mapsto \{p\}_{\prime}$ is an order isomorphism from the projections of $Z(A)$ onto the split faces of $S_*(A)$.

A self-adjoint element s of a JBW*-algebra A is said to be a *symmetry* if s^2 is the unit. By [51], 3.2.10, $Q(s)^2$ is the identity map on A and by Macdonald's theorem, $Q(s)$ is an automorphism of A_{sa}. In particular, $Q(s)$ maps projections to projections.

2.2.2 JB*-triples

Let B be a unital Banach algebra with unit 1 and let B_1^* be the dual unit ball of B. The *state space* $S(B)$ of B is defined by

$$S(B) = \{x \in B_1^* : x(1) = 1\};$$

and the numerical range of an element T in B is the subset $V(T)$ of the complex plane defined by

$$V(T) = \{x(T) : x \in S(B)\}.$$

The element T in B is said to be *hermitian* if the numerical range is a subset of the real line.

Let A be a Jordan *-triple equipped with a norm with respect to which it is a Banach space and let $B(A)$ be the Banach algebra of bounded operators on A. Then A is said to be a *J*-triple* if:

(i) for all a, b and c in A, the linear operator $D(a,b)c \mapsto \{a\,b\,c\}$ lies in $B(A)$;

(ii) the map $D : A \times A \mapsto B(A)$ given by $(a,b) \mapsto D(a,b)$ is continuous;

(iii) for all a in A, the linear operator $D(a,a)$ is an hermitian element of $B(A)$.

Since D is a sesquilinear form, (ii) is equivalent to requiring that D is bounded. This shows that the norm-closure of a sub-triple of a J*-triple A is also a sub-triple of A. The J*-triple A is said to be *positive* if $D(a,a)$ is a positive element of $B(A)$. The following result is [58], 5.3.

Proposition 2.11 *Let A be a J*-triple. For elements a and b in A, let $D(a)$ be the map $D(a) : b \mapsto \{a\,a\,b\}$ and let $A(a)$ denote the smallest norm-closed subtriple of A containing a. Then, the following conditions are equivalent:*

(i) *A is positive and $\|D(a)\|$ equals $\|a\|^2$ for all a in A;*

(ii) *A is positive and $\|\{a\,a\,a\}\|$ equals $\|a\|^3$ for all a in A;*

(iii) *for every a in A, there exists an isometric triple isomorphism from $A(a)$ onto a commutative C*-algebra.*

A J*-triple satisfying the equivalent conditions of Proposition 2.11 is said to be a *JB*-triple*. JB*-triples were introduced by Kaup in [58] (although they had previously been discussed under the name C*-triples in [57]). It follows from the Gelfand-Naimark Theorem for JB*-triples ([45]) that, for all a, b and c in A,

$$\|\{a\,b\,c\}\| \leq \|a\|\|b\|\|c\|.$$

JB*-algebras are JB*-triples with the natural triple product. In particular C*-algebras are JB*-triples.

Let A be a JB*-triple. A linear functional x on A is said to be a *character* of A, if it is a non-zero triple homomorphism from A to $\mathbb{C}$. The set $\Delta(A)$ of characters of A, equipped with the w*-topology, is said to be the *character space* of A.

Theorem 2.12 is essential to the theory of JB*-triples.

Theorem 2.12 *Let A be a JB*-triple with triple character space $\Delta(A)$. For an element a of A, let $A(a)$ be the smallest JB*-subtriple of A containing a, let $\sigma(a)$ be the set*

$$\sigma(a) = \{x(a) : x \in \Delta(A)\} \cap \mathbb{R}^+$$

with the locally compact topology induced from $\mathbb{C}$ and let $C_0(\sigma(a))$ be the commutative C-algebra of continuous complex-valued functions vanishing at infinity on $\sigma(a)$. Then there exists an isometric triple isomorphism $\phi : A(a) \mapsto C_0(\sigma(a))$ mapping a to the positive generating function $\iota : \lambda \mapsto \lambda$ of $C_0(\sigma(a))$.*

For each element a of the JB*-triple A, the space $\sigma(a)$ defined in Theorem 2.12 is said to be the *spectrum* of a in A. The inverse of ϕ, taking f in $C^b(\sigma(a))$ to an element $f(a)$ of $A(a)$ is said to be the *functional calculus* of a.

The triple product and the norm of a JB*-triple are intimately connected, as witnessed by Theorem 2.13 ([58], Section 5).

Theorem 2.13 *Let A and B be JB*-triples and let $\Phi : A \mapsto B$ be a linear surjection. Then Φ is an isometry if and only if Φ is a triple isomorphism.*

2.2.3 JBW*-triples

A JB*-triple which is the dual of a complex Banach space is said to be a *JBW*-triple*. Let A be a JBW*-triple with predual A_*. The elements of A_* are said to be *normal functionals*. Fundamental to the theory of JBW*-triples is Theorem 2.14, proved in [9], Theorem 2.1.

Theorem 2.14 *Let A be a JBW*-triple. Then the predual is unique up to isometric isomorphism and the triple product is separately w*-continuous.*

Corollary 2.15 *Let A be a JBW*-triple with predual A_*, let u be a tripotent in A and let $\{P_j(u) : j = 0, 1, 2\}$ be the Peirce projections associated with u, with corresponding Peirce spaces $\{A_j(u) : j = 0, 1, 2\}$. Then the Peirce projections are w*-continuous and the adjoints of unique idempotents $\{P_j(u)_* : j = 0, 1, 2\}$ on A_*. For j equal to 0, 1, 2, the Peirce space $A_j(u)$ is a JBW*-triple, and the predual $A_j(u)_*$ may be considered as a subspace of A_* via the identification:*

$$A_j(u)_* \cong P_j(u)_*A_* = \{y \in A_* : y = P_j(u)_*y\}.$$

In particular, $A_2(u)$ is a JBW-algebra under the multiplication and involution defined in Theorem 2.3.*

Related to Theorem 2.14 is Theorem 2.16, proved in [23], Corollary 11.

Theorem 2.16 *The second dual of a JB*-triple is a JBW*-triple.*

The next theorem, [44], Proposition 2 is critical to the study and classification of normal functionals on JBW*-triples, and hence to functionals on JB*-triples.

Theorem 2.17 *Let A be a JBW*-triple and let x be an element of A_*. Then there exists a unique tripotent $e(x)$ in A with Peirce-2 projection $P_2(e(x))$ and Peirce-2 space $A_2(e(x))$ such that x agrees with $P_2(e(x))_*x$, and the restriction of x to $A_2(e(x))$ is a faithful normal positive functional on $A_2(e(x))$.*

The tripotent $e(x)$ defined in Theorem 2.17 is said to be the *support tripotent* for x. Corollary 2.18, the well-known polar decomposition theorem for normal functionals of a W*-algebra, is deduced from Theorem 2.17 in [46], Theorem 2.12.

Corollary 2.18 *Let A be a W*-algebra and let x be a normal functional on A. Then there exists a unique pair $(|x|, e(x))$ where $|x|$ is a positive normal functional such* *that x and $|x|$ have the same norm, for all elements a in A,*

$$|x|(a) = x(e(x)a)$$

and $e(x)^*e(x)$ is the support of $|x|$.

The positive normal functional $|x|$ defined in Corollary 2.18 is said to be the *absolute value* of x. When x is positive, x and $|x|$ agree and $e(x)$ is the support projection.

Let A be a JBW*-triple with predual A_* and let $\partial_e A_{*,1}$ be the set of extreme points of $A_{*,1}$, the unit ball of A_*. The elements of $\partial_e A_{*,1}$ are said to be the *pure normal functionals* of A. Let A be a JB*-triple with dual A^* and bi-dual A^{**} and let $\partial_e A_1^*$ be the set of extreme points of the dual unit ball A_1^*. The elements of $\partial_e A_1^*$ are said to be the *pure functionals* of A. Clearly the set of pure functionals of A coincides with the set of normal pure functionals of A^{**}. Pure functionals are fundamental to investigating the structure of JB*-triples (see, for example, [44], [45] and [20]). Their importance will become apparent in subsequent chapters.

Theorem 2.19 is another crucial result from [44].

Theorem 2.19 *Let A be a JBW*-triple, let $\partial_e A_{*,1}$ be the set of pure normal functionals of A, and for x in $\partial_e A_{*,1}$, let $e(x)$ be the support tripotent of x. Then, the map $x \mapsto e(x)$ is a bijection from $\partial_e A_{*,1}$ onto the set of minimal tripotents of A.*

A JBW*-triple containing no non-trivial proper w*-closed ideals and a minimal tripotent is said to be a *Cartan factor*.

Theorem 2.20 is the main result of [33].

Theorem 2.20 *Let A be a JBW*-triple, let $\mathcal{U}(A)$ be the partially ordered set of tripotents of A and let $A_{*,1}$ be the unit ball of the predual of A. For u in $\mathcal{U}(A)$, define*

$$\{u\}_{\prime} = \{x \in A_{*,1} : x(a) = 1\}.$$

Then the map $u \mapsto \{u\}_{\prime}$ is an order isomorphism from $\mathcal{U}(A)$ onto the set of proper norm-closed faces of $A_{*,1}$, partially ordered by set inclusion.

Theorem 2.21 can be found in [44] and [33].

Theorem 2.21 *Let A be a JBW*-triple and let a be an element of unit norm in A. Let $W(a)$ be the JBW*-subtriple of A generated by a. Then there exists a unique tripotent $r(a)$ in $W(a)$, with Peirce 2-space $A_2(r(a))$ such that a lies in $A_2(r(a))_1^+$, the positive part of the unit ball in A. Furthermore, the JBW*-subalgebra of $A_2(r(a))$ generated by a has unit $r(a)$ and coincides with $W(a)$ as a JB*-triple.*

For an element a of a JBW*-triple A, the tripotent $r(a)$ defined by Theorem 2.21 is said to be the *support tripotent* of a in A. The following result may be found in [20], Section 2.

Proposition 2.22 *Let A be a JB*-triple, let a be an element of A, let $r(a)$ be the support tripotent of a in A^{**} and let $A_2^{**}(r(a))$ be the Peirce-2 space of $r(a)$ in A^{**}. Let $I_n(a)$ be the smallest norm-closed inner ideal of A containing a, and let $I_n(a)^{**}$ be its bi-dual. Then,*

$$I_n(a) = \overline{\{a\, A\, a\}}^n,$$

$I_n(a)$ is a JB-subalgebra of $A_2^{**}(r(a))$, containing a as a positive element and*

$$I_n(a)^{**} \cong A_2^{**}(r(a)).$$

Chapter 3

Lattice theory and M-structure

As in any algebraic structure, the properties of the set of ideals of a Jordan*-triple are of fundamental importance. Because of the interplay between topology and algebraic structure, the complete lattice of norm-closed ideals of a JB*-triple and the Boolean algebra of w*-closed ideals of a JBW*-triple demand particular attention. Indeed, closed ideals can be characterised among the subspaces purely in terms of the norm, a fact which leads us to the investigation of M-ideals and M-summands in a general Banach space. Topological spaces of norm-closed ideals will play a central role in Chapter 5 and Chapter 6, as base spaces over which JB*-triples may be represented as cross-sections. Norm-closed ideals satisfying certain lattice theoretic properties and their relation to certain spaces of functionals will be studied in Chapter 7.

Many of the concepts and results required for the subsequent investigation of JB*-triples and their ideals rely on purely Banach space or order theoretic considerations. In this chapter such concepts will be reviewed in the general setting. In Chapter 4 the discussion will continue by specialising to JB*-triples in order to obtain further results which require the additional structure afforded by the algebraic and geometric properties of JB*-triples.

In Section 3.1, basic definitions and results from the M-structure of Banach spaces are recalled. With the complete lattice of M-ideals of a Banach space as a motivational example, lattice theoretic concepts fundamental to the thesis are introduced in Section 3.2. In Section 3.3, the set of pure functionals and Banach space spectra are introduced. It is shown that the primitive spectrum is an example of a primitive subset of the complete lattice of M-ideals of a Banach space, allowing the lattice theory developed in Section 3.2 to be applied to Banach spaces. The discussion of M-structure in arbitrary Banach spaces is concluded in section 3.4 with an alternative description of Banach space spectra in terms of factor representations.

3.1 Summands and M-ideals

In this section some definitions and results from the M-structure theory of Banach spaces are recalled. For details, the reader is referred to [1], [2], [13] and [52]. Let A be a Banach space. A closed subspace J of A is said to be an *M-summand* if there exists a closed subspace $J^{\perp}$ such that every element a of A possesses a unique decomposition

$$a = b + b^{\perp}$$

with b in J and $b^{\perp}$ in $J^{\perp}$, satisfying the condition

$$\|a\| = \max\{\|b\|, \|b^{\perp}\|\}.$$

A closed subspace J of A is said to be an *L-summand* if there exists a closed subspace $J^{\perp}$ such that every element a of A possesses a unique decomposition

$$a = b + b^{\perp}$$

with b in J and $b^{\perp}$ in $J^{\perp}$, satisfying the norm condition

$$\|a\| = \|b\| + \|b^{\perp}\|.$$

If J is an M-summand or L-summand of A then the uniquely determined subspace $J^{\perp}$ is said to be the *complement* to J and is respectively an M-summand or L-summand of A. The complementary spaces A and $\{0\}$ are said to be the *trivial* M-summands and L-summands.

A projection P in the algebra $B(A)$ of bounded linear operators on A is said to be an *M-projection* if, for all elements a in A,

$$\|a\| = \max\{\|Pa\|, \|(I-P)a\|\}$$

and is said to be an *L-projection* if, for all elements a in A,

$$\|a\| = \|Pa\| + \|(I-P)a\|.$$

Theorem 3.1 *Let A be a Banach space and let $P_M(A)$ and $P_L(A)$ respectively denote the sets of M-projections, L-projections on A. Then, for P and Q in $P_M(A)$, respectively, P and Q in $P_L(A)$, a partial order and complementation are defined on $P_M(A)$, respectively $P_L(A)$, by:*

$$P \leq Q \Leftrightarrow PQ = P;$$
$$P^{\perp} = I - P.$$

With respect to this ordering, $P_M(A)$ is a lattice and $P_L(A)$ is a complete lattice, with the meet and join operations satisfing the equations:

$$P \wedge Q = PQ;$$
$$P \vee Q = P + Q - PQ.$$

With respect to these operations, $P_M(A)$ is a Boolean algebra and $P_L(A)$ is a complete Boolean algebra.

A close relationship between M-projections and M-summands and L-projections and L-summands is established in Corollary 3.2. The result is an amalgamation of [13], Lemma 1.4, [13], Corollary 1.8 and [13], 1.10.

Corollary 3.2 *Let A be a Banach space and let J and K be respectively M-summands or L-summands of A. Then a partial order and complementation are defined on the* *set of M-summands of A, respectively L-summands of A, by:*

$$J \leq K \Leftrightarrow J \subseteq K;$$
$$J \mapsto J^{\perp}.$$

With respect to this ordering, the set of M-summands of A forms a lattice and the set of L-summands of A forms a complete lattice, with the meet and join operators satisfing the equations:

$$J \wedge K = J \cap K;$$
$$J \vee K = J + K.$$

With respect to these operations, the set of M-summands of A and the set of L-summands of A form Boolean algebras. The Boolean algebra of L-summands is complete, and, for any collection $\{L_\lambda : \lambda \in \Lambda\}$ *of L-summands,*

$$\begin{aligned} \bigwedge\{L_\lambda : \lambda \in \Lambda\} &= \bigcap\{L_\lambda : \lambda \in \Lambda\}, \\ \bigvee\{L_\lambda : \lambda \in \Lambda\} &= \overline{\mathrm{lin}\{L_\lambda : \lambda \in \Lambda\}}^{n}. \end{aligned}$$

The map $P \mapsto \mathrm{range}\, P$ *is an ortho-order isomorphism from* $P_M(A)$ *onto the Boolean algebra of M-summands and from* $P_L(A)$ *onto the complete Boolean algebra of L-summands, and the mapping* $P \mapsto \ker P$ *is an anti-order isomorphism from* $P_M(A)$ *onto the Boolean algebra of M-summands and from* $P_L(A)$ *onto the complete Boolean algebra of L-summands such that*

$$\mathrm{range}\, P = (\ker P)^{\perp}, \qquad \mathrm{range}\, P^{\perp} = \ker P.$$

For a general Banach space, the Boolean algebras of M-projections and M-summands may not be complete ([13], Theorem 1.10).

M-ideals are defined and studied using the theory of topological annihilators, recalled in Lemma 3.3 ([65], Theorem 4.9). The existence of an isometric isomorphism between two Banach spaces E and F will be denoted $E \cong F$.

Lemma 3.3 *Let A be a Banach space, with dual A^* and bidual A^{**}, let $\iota : A \mapsto A^{**}$ be the canonical embedding, let M be a subspace of A, and let L be a subspace of A^*. Define the set M° of A^* by*

$$M^\circ = \{x \in A^* : x|_M = 0\},$$

and the set $L_\circ$ of A by

$$L_\circ = \{a \in A : \iota(a)|_L = 0\}.$$

Then M° is a w-closed subspace of A^*, $L_\circ$ is a norm-closed subspace of A, and*

$$(M^\circ)_\circ = \overline{M}^n, \qquad (L_\circ)^\circ = \overline{L}^{w*}, \qquad \iota(M)_\circ = M^\circ.$$

When M is a norm-closed subspace of A,

$$M^* \cong \frac{A^*}{M^\circ}, \qquad \left(\frac{A}{M}\right)^* \cong M^\circ.$$

and

$$\overline{\iota(M)}^{w*} = M^{\circ\circ} \cong \left(\frac{A^*}{M^\circ}\right)^* \cong M^{**}.$$

In the situation described in Lemma 3.3, M° and $L_\circ$ are said to be the *topological annihilators* of M in the weak topology and L in the w*-topology respectively.

A subspace J of A is said to be an *M-ideal* if J° is an L-summand. The spaces A and $\{0\}$ are said to be the trivial M-ideals.

Theorem 3.4 *Let A be a Banach space. The set of M-ideals forms a complete lattice when ordered by set inclusion. For a finite collection of M-ideals $J_1, \ldots, J_n$,*

$$\bigwedge_{j=1}^{n} J_j = \bigcap_{j=1}^{n} J_j.$$

For an arbitrary collection $\{J_\lambda : \lambda \in \Lambda\}$ of M-ideals,

$$\bigvee\{J_\lambda : \lambda \in \Lambda\} = \overline{\mathrm{lin}\{J_\lambda : \lambda \in \Lambda\}}^{n}.$$

Note that the meet of a non-finite collection of M-ideals of a Banach space may be strictly contained in the intersection of the collection.

Proposition 3.5 *Let A be a Banach space with dual A^*, let $P_M(A)$ be the Boolean algebra of M-projections of A, let $P_L(A)$ be the complete Boolean algebra of L-projections of A, let $P_M(A^*)$ be the Boolean algebra of M-projections of A^* and let $P_L^{w*}(A^*)$ be the set of w*-continuous L-projections ordered by set inclusion. Then:*

(i) *every element of $P_M(A^*)$ is w*-continuous;*

(ii) *the set of M-summands of A^* and the set of w*-closed M-ideals of A^* coincide;*

(iii) *$P_M(A^*)$ is a complete Boolean algebra;*

(iv) *the map $P \mapsto P^*$ is an order isomorphism from $P_L(A)$ onto $P_M(A^*)$ and from $P_M(A)$ onto $P_L^{w*}(A^*)$;*

(v) *the map $J \mapsto J^\circ$ is an anti-order isomorphism from the Boolean algebra of L-summands on A onto the Boolean algebra of M-summands on A^* and from the Boolean algebra of M-summands on A into the complete lattice of w*-closed L-summands on A^* such that*

$$(\mathrm{range}\, P)^\circ = (\mathrm{range}\, P^*)^\perp, \qquad (\mathrm{range}\, P^\perp)^\circ = \mathrm{range}\, P^*;$$

(vi) *the map $J \mapsto J^\circ$ is an anti-order isomorphism from the complete lattice of M-ideals of A onto the complete lattice of w*-closed L-summands of A^*.*

Lemma 3.6 can be found in [13], Proposition 1.18, Proposition 2.9.

Lemma 3.6 *Let A be a Banach space and let J be respectively an M-summand, L-summand, M-ideal of A. Then, the following results hold.*

(i) *A subspace K of J is an M-summand, L-summand, M-ideal respectively of J if and only if it is an M-summand, L-summand, M-ideal respectively of A contained in J.*

(ii) *A subspace K of A/J is an M-summand, L-summand, M-ideal respectively of of A/J if and only if it is the image of an M-summand, L-summand, M-ideal respectively of A under the quotient map.*

When J is an M-summand or an L-summand with complement $J^\perp$,

$$J^\perp \cong A/J.$$

Lemma 3.7 reveals the useful fact that both the topological annihilator of an M-ideal or L-summand and its complement can be considered as dual spaces in a natural way.

Lemma 3.7 *Let A be a Banach space with dual A^*, and let J be an M-ideal or an L-summand of A with topological annihilator J°. Then the complementary summands J° and $(J^\circ)^\perp$ of A^* have the natural identifications,*

$$J^\circ \cong (A/J)^*, \qquad (J^\circ)^\perp \cong A^*/J^\circ \cong J^*.$$

under which, the w-topology of $(A/J)^*$ coincides with the relative w*-topology of A^* on J° and the w*-topology of J^* is weaker than the relative w*-topology of A^* on $(J^\circ)^\perp$. When J is an M-ideal,*

$$(J^\circ)^\perp = \{x \in A^* : \|x\| = \|x|J\|\}.$$

Finally, Lemma 3.8 shows that the set of extreme points of the unit ball in a Banach space decomposes over M-structure.

Lemma 3.8 *Let A be a Banach space with dual A^*, let L and $L^\perp$ be complementary L-summands of A, and let $\partial_e A_1$, $\partial_e L_1$ and $\partial_e L_1^\perp$ be the set of extreme points of the unit ball of A, L and $L^\perp$ respectively. Let J_1 and J_2 be M-ideals of A with topological* *annihilators J_1°, J_2° respectively. Then $\partial_e A_1$ has the decomposition into disjoint sets*

$$\partial_e A_1 = \partial_e L_1 \cup \partial_e L_1^\perp,$$

$\partial_e L_1$ is the set $\partial_e A_1 \cap L$ and

$$(J_1 \cap J_2)^\circ \cap \partial_e A_1^* = (J_1^\circ \cap \partial_e A_1^*) \cup (J_2^\circ \cap \partial_e A_1^*). \tag{3.1.1}$$

3.2 Lattices

Let A be a Banach space, let $\mathcal{ZI}_{\rm n}(A)$ denote the complete lattice of M-ideals of A, and let $\mathcal{ZI}(A)$ be the Boolean algebra of M-summands of A. Various subsets of $\mathcal{ZI}_{\rm n}(A)$ will play an important role in the sequel, particularly in the search for a base-space for JB*-triples. These subsets will be equipped with topologies defined in terms of the lattice structure, and the ideals contained in these subsets will satisfy certain lattice theoretic properties. For clarity, these topologies and properties are introduced in a purely lattice theoretic setting. The reader is referred to [47] for a detailed account of the lattice theory involved.

Let $\mathcal{L}$ be a lattice with least element 0 and let P be an element of $\mathcal{L}$. The element P is said to be *prime* if whenever J and K are elements of $\mathcal{L}$ such that $J \wedge K \leq P$, then $J \leq P$ or $K \leq P$. The element P is said to be *primal* if for any elements $J_1, \ldots, J_n$ in L such that $J_1 \wedge \ldots \wedge J_n$ is equal to 0, there exists an element j in $1, \ldots, n$ such that $J_j \leq P$. Let $\mathrm{Prime}\,\mathcal{L}$ and $\mathrm{Primal}\,\mathcal{L}$ respectively denote the sets of prime and primal elements of $\mathcal{L}$. By an inductive argument, it can be seen that every prime element is primal.

Lemma 3.9 *Let $\mathcal{L}$ be a Boolean algebra and let P be an element of $\mathcal{L}$. Then the following are equivalent:*

(i) *P is prime;*

(ii) *P is primal;*

(iii) P *is maximal or* 1*;*

(iv) $P^{\perp}$ *is minimal or* 0.

Proof. (i)⇒(ii) This is true for all complete lattices.

(ii)⇒(iii) Let P be primal and let K in $\mathcal{L}$ be such that $P \leq K$. Then $K \wedge K^{\perp}$ is zero, and either $K \leq P$, in which case K and P coincide, or $K^{\perp} \leq P$, in which case $K^{\perp} \leq K$ and K is 1. Thus P is maximal or 1.

(iii)⇒(iv) This is obvious.

(iv)⇒(i) If $P^{\perp}$ is 0 then P is 1 and hence prime. Let $P^{\perp}$ be minimal and let J and K be elements of $\mathcal{L}$ such that $J \wedge K \leq P$. Assume that $P^{\perp} \leq J$ and $P^{\perp} \leq K$. Then $P^{\perp} \leq P$ which implies that $P^{\perp}$ is 0, a contradiction. Without loss of generality, assume that $P^{\perp} \not\leq J$. Then $P^{\perp} \wedge J \neq P^{\perp}$ and therefore $P^{\perp} \wedge J$ is 0. This implies that $J \leq P$ and therefore P is prime. ■

For a subset $\mathcal{S}$ of the lattice $\mathcal{L}$ and for an element J in $\mathcal{L}$, write

$$\downarrow \mathcal{S} = \{K \in \mathcal{L} : K \leq L \text{ for some } L \in S\}, \qquad \downarrow J = \downarrow \{J\},$$
$$\uparrow \mathcal{S} = \{K \in \mathcal{L} : K \geq L \text{ for some } L \in S\}, \qquad \uparrow J = \uparrow \{J\}.$$

We say $\mathcal{S}$ is *lower* if it coincides with $\downarrow \mathcal{S}$ and *upper* if it coincides with $\uparrow \mathcal{S}$.

For every finite subset $\mathcal{F}$ of $\mathcal{L}$, define

$$\mathcal{U}(\mathcal{F}) = \{K \in \mathcal{L} : J \not\leq K \quad \forall J \in \mathcal{F}\},$$

$$\mathcal{U}(J) = \mathcal{U}(\{J\}) = \mathcal{L} \setminus \uparrow J.$$

The *lower topology* $\mathfrak{W}$ of $\mathcal{L}$ is the topology generated by the sub-basis

$$\{\mathcal{U}(J) : J \in \mathcal{L}\}.$$

The set

$$\{\mathcal{U}(\mathcal{F}) : \mathcal{F} \subseteq \mathcal{L},\ \mathcal{F} \text{ finite}\}$$

forms a basis for $\mathfrak{W}$. Observe that every $\mathfrak{W}$-open set is lower and, for any element J in $\mathcal{L}$,

$$\overline{\{J\}}^{\mathfrak{W}} = \{K \in \mathcal{L} : J \leq K\} = \uparrow J.$$

When $\mathcal{L}$ is a complete lattice, for any subset $\mathcal{S}$ of $\mathcal{L}$,

$$\bigcup_{J \in \mathcal{S}} \mathcal{U}(J) = \mathcal{U}(\vee S). \tag{3.2.1}$$

Let $\mathcal{L}$ be a complete lattice and let $\mathfrak{S}$ be the family of upper subsets $\mathcal{U}$ of $\mathcal{L}$ such that for all directed subsets $\mathcal{D}$ of $\mathcal{L}$ for which $\sup \mathcal{D}$ lies in $\mathcal{U}$ it follows that $\mathcal{D} \cap \mathcal{U}$ is non-empty. Then $\mathfrak{S}$ forms a topology for $\mathcal{L}$, known as the *Scott topology*. A subset $\mathcal{U}$ of $\mathcal{L}$ lies in $\mathfrak{S}$ if and only if it is upper, and, for any subset $\mathcal{S}$ of $\mathcal{L}$ such that $\vee\mathcal{S}$ is an element of $\mathcal{U}$, there exists a finite subset $\mathcal{F}$ of $\mathcal{S}$ such that $\vee\mathcal{F}$ is an element of $\mathcal{U}$.

For any element J in $\mathcal{L}$,

$$\overline{\{J\}}^{\mathfrak{S}} = \{K \in \mathcal{L} : J \geq K\} = \downarrow J.$$

The third topology for the complete lattice $\mathcal{L}$ introduced in this section is the *Lawson topology* $\mathfrak{L}$, which is defined to be the intersection of the lower and Scott topologies.

The lower topology $\mathfrak{W}$ has a particularly simple form on subsets of $\operatorname{Prime}\mathcal{L}$. Let $\mathcal{T}$ be a subset of $\operatorname{Prime}\mathcal{L}$, let $\mathfrak{W}_{\mathcal{T}}$ denote the relative topology induced on $\mathcal{T}$ by $\mathfrak{W}$ and, for J in $\mathcal{L}$, define

$$\mathcal{U}_{\mathcal{T}}(J) = \mathcal{U}(J) \cap \mathcal{T} = \{P \in \mathcal{T} : J \not\leq P\}.$$

For any finite subset $\mathcal{F}$ of $\mathcal{L}$, the equation

$$\bigcap_{J \in \mathcal{F}} \mathcal{U}_{\mathcal{T}}(J) = \mathcal{U}_{\mathcal{T}}(\wedge\mathcal{F}).$$

supplements equation (3.2.1). Thus, $\mathfrak{W}_{\mathcal{T}}$ is exactly the set of subsets of $\mathcal{T}$ of the form

$\mathcal{U}_{\mathcal{T}}(J)$ for some element J in $\mathcal{L}$. Denote by $\mathrm{hull}_{\mathcal{T}}\, J$ the set of $\mathfrak{W}_{\mathcal{T}}$-closed complements

$$\mathrm{hull}_{\mathcal{T}}\, J = \{P \in \mathcal{T} : J \leq P\} = \mathcal{T} \backslash \mathcal{U}(J).$$

Let $\mathcal{L}$ be a complete lattice. Then, a subset $\mathcal{T}$ of $\mathcal{L}$ is said to *order generate* $\mathcal{L}$ if

$$\mathcal{L} = \{\wedge S : S \subseteq \mathcal{T}\}.$$

A subset $\mathcal{T}$ of $\mathrm{Prime}\,\mathcal{L}$ is said to be *primitive* if it order generates $\mathcal{L}$. The motivating example for this definition is the primitive spectrum of a Banach space, which will be introduced in Section 3.3.

Proposition 3.10 *Let $\mathcal{L}$ be a complete lattice, let $\mathcal{T}$ be a primitive subset of $\mathcal{L}$ and let $\mathfrak{W}_{\mathcal{T}}$ be the relativised lower topology of $\mathcal{T}$. Then, the mapping $J \mapsto \mathrm{hull}_{\mathcal{T}}\, J$ is an anti order-isomorphism from $\mathcal{L}$ onto the set of $\mathfrak{W}_{\mathcal{T}}$-closed subsets of $\mathcal{T}$, with inverse $C \mapsto \wedge C$ and, for each subset $\mathcal{S}$ of $\mathcal{T}$ with $\mathfrak{W}_{\mathcal{T}}$-closure $\overline{\mathcal{S}}^{\mathfrak{W}_{\mathcal{T}}}$,*

$$\overline{\mathcal{S}}^{\mathfrak{W}_{\mathcal{T}}} = \mathrm{hull}_{\mathcal{T}} \wedge \mathcal{S}.$$

Proof. Clearly, the mapping $J \mapsto \mathrm{hull}_{\mathcal{T}}\, J$ is surjective. Let J be an element of $\mathcal{L}$. Since $\mathcal{T}$ is order generating, J is of the form $\wedge \mathcal{S}$ for some subset $\mathcal{S}$ of $\mathcal{T}$, and $\mathcal{S}$ is a subset of $\mathrm{hull}_{\mathcal{T}}\, J$. Thus,

$$\bigwedge \mathrm{hull}_{\mathcal{T}}\, J \leq \wedge \mathcal{S} = J \leq \bigwedge \mathrm{hull}_{\mathcal{T}}\, J,$$

and the mapping $C \mapsto \wedge C$ is an inverse for the mapping $J \mapsto \mathrm{hull}_{\mathcal{T}}\, J$. Clearly both maps are order reversing. Let C be a $\mathfrak{W}_{\mathcal{T}}$ closed subset of $\mathcal{T}$ containing S. Then $\wedge S$ dominates $\wedge C$ and

$$\mathcal{S} \subseteq \mathrm{hull}_{\mathcal{T}} \wedge \mathcal{S} \subseteq \mathrm{hull}_{\mathcal{T}} \wedge C = C.$$

The result follows. ■

Theorem 3.11 *Let $\mathcal{L}$ be a complete lattice, let $\mathcal{T}$ be a primitive subset of $\mathcal{L}$, and let $\mathfrak{W}$ be the lower topology on $\mathcal{L}$. Then,*

$$\operatorname{Primal}\mathcal{L} = \overline{\operatorname{Prime}\mathcal{L}}^{\mathfrak{W}} = \overline{\mathcal{T}}^{\mathfrak{W}}.$$

Proof. The result follows by the argument of [53], Theorem 6.9. ■

Proposition 3.12 *Let $\mathcal{L}$ be a complete lattice, and let $\mathcal{T}$ be a primitive subset of $\mathcal{L}$. Then, for each element J in $\mathcal{L}$, the following are equivalent:*

(i) *J is primal;*

(ii) *if $J_1, \ldots, J_n$ are elements of $\mathcal{L}$ such that, for $j = 1, \ldots, n$, $J_j \not\leq J$ then $\bigwedge_{j=1}^{n} J_j$ is non-zero;*

(iii) *if $U_1, \ldots, U_n$ are $\mathfrak{W}_{\mathcal{T}}$-open subsets of $\mathcal{T}$ which intersect* $\operatorname{hull}_{\mathcal{T}} J$ *then $\bigcap_{j=1}^{n} U_j$ is non-empty;*

(iv) *there is a net $(P_\lambda)_{\lambda\in\Lambda}$ in $\mathcal{T}$ that is $\mathfrak{W}$-convergent to every point of* $\operatorname{hull}_{\mathcal{T}} J$.

Proof. The proof can be found in [6], Proposition 3.2. ■

It will be shown in Chapter 5 that if a JB*-triple possesses a base space of primal ideals with the properties required for a Gelfand representation to exist then the base space is the space of minimal primal ideals with the lower topology, and that this space may be constructed from the primitive spectrum by means of a process known as complete regularisation. The remainder of this section develops the lattice theoretic part of that theory.

A topological space Y is said to be *completely regular* if it is Hausdorff, and, for every closed subset C and every element P in $Y\backslash C$, there exists a continuous function f such that f is equal to zero on C and to 1 at P [48].

Proposition 3.13 *Let (X, τ) be a topological space, let $C^b(X)$ be the space of bounded continuous complex-valued functions on X and let $\approx$ be the equivalence relation on*

*X defined, for elements P and Q of X, by $P \approx Q$ if and only if, for every element f in $C^b(X)$, $f(P)$ coincides with $f(Q)$. For each element P in X, let $[P]$ denote the equivalence class of P, let γX denote the set of equivalence classes, let $\gamma : X \mapsto \gamma X$ be the natural quotient map, and for each element f of $C^b(X)$, define the function $[f]$ on γX by $[f]([P]) = f(P)$. Let $\mathfrak{Q}$ be the quotient topology induced on γX by γ and let $\mathfrak{CR}$ be the weakest topology for which for all elements f of $C^b(X)$, $[f]$ is continuous. Then $\mathfrak{CR}$ is a completely regular topology on γX, weaker than $\mathfrak{Q}$, with equality holding when X is compact. The map $f \mapsto [f]$ is an isometric *-isomorphism from $C^b(X)$ onto $C^b(\gamma X)$, the space of bounded continuous complex-valued functions on γX.*

For any topological space X, the construction $(\gamma, \gamma X)$ of Proposition 3.13 is said to be the *complete regularisation* of X.

Proposition 3.14 *Let X be a topological space with complete regularisation $(\gamma, \gamma X)$ and let π be a continuous mapping of X into a completely regular space Y. Then there exists a continuous map $\pi' : \gamma X \mapsto Y$ such that*

$$\pi = \pi' \circ \gamma.$$

For more information about complete regularisation, see [48], [76].

Proposition 3.15 *Let $\mathcal{L}$ be a complete lattice possessing a primitive subset $\mathcal{T}$, let* Primal $\mathcal{L}$ *be the set of primal elements of $\mathcal{L}$, let* MinPrimal $\mathcal{L}$ *be the set of minimal elements of* Primal $\mathcal{L}$*, let $(\gamma, \gamma\mathcal{T})$ be the complete regularisation of $\mathcal{T}$ with respect to the lower topology $\mathfrak{W}$, let the quotient topology on $\gamma\mathcal{T}$ be denoted by $\mathfrak{Q}$ and let $\Omega(\mathcal{L})$ be the image of $\gamma\mathcal{T}$ in $\mathcal{L}$ under the map $p : S \mapsto \wedge S$. Then the following results hold.*

(i) *γ is $\mathfrak{W}$ to $\mathfrak{Q}$ open if and only if p is $\mathfrak{Q}$ to $\mathfrak{W}$ continuous.*

(ii) *The set $\Omega(\mathcal{L})$ is a subset of* Primal $\mathcal{L}$ *if and only if $\Omega(\mathcal{L})$ and* MinPrimal $\mathcal{L}$ *coincide as sets. When either condition holds, p is a $\mathfrak{Q}$ to $\mathfrak{W}$ open map.*

Proof. (i) Let J be an element of $\mathcal{L}$ and let $\mathcal{U}(J)$ be the subset

$$\mathcal{U}(J) = \{K \in \mathcal{L} : J \subsetneq K\}.$$

Let P in $\mathcal{T}$ be such that $\gamma(P)$ lies in $p^{-1}(\mathcal{U}(J) \cap \Omega(\mathcal{L}))$. Then, there exists an element Q in $\gamma(P)$ such that Q lies in $\mathcal{U}(J) \cap \mathcal{T}$. Then $\gamma(P)$ coincides with $\gamma(Q)$ and lies in $\gamma(\mathcal{U}(J) \cap \mathcal{T})$. Thus

$$p^{-1}(\mathcal{U}(J) \cap \Omega(\mathcal{L})) \subseteq \gamma(\mathcal{U}(J) \cap \mathcal{T}).$$

Conversely, let P lie in $\mathcal{U}(J) \cap \mathcal{T}$. Then $p\gamma(P)$ is dominated by P, and, hence, $p\gamma(P)$ cannot dominate J. Therefore, $\gamma(P)$ lies in $p^{-1}(\mathcal{U}(J) \cap \Omega(\mathcal{L}))$. This implies that,

$$p^{-1}(\mathcal{U}(J) \cap \Omega(\mathcal{L})) \supseteq \gamma(\mathcal{U}(J) \cap \mathcal{T}).$$

Since the sets $\mathcal{U}(J)$ form a sub-basis for the lower topology on $\mathcal{L}$, the result follows.

(ii) This follows by the argument of [7], Lemma 3.1. ■

Lemma 3.16 *Let $\mathcal{L}$ be a complete lattice, let* Primal $\mathcal{L}$ *be the set of primal elements of $\mathcal{L}$ and let* MinPrimal $\mathcal{L}$ *be the set of minimal primal elements of $\mathcal{L}$. Then every element of* Primal $\mathcal{L}$ *contains an element of* MinPrimal $\mathcal{L}$.

Proof. Let C be a totally ordered subset of Primal $\mathcal{L}$. Assume that $\wedge C$ is not primal. Then, there exist elements $J_1, \ldots, J_n$ in $\mathcal{L}$ such that $\wedge_{j=1}^n J_j$ is zero but for j equal to $1, \ldots, n$, $J_j \not\leq \wedge C$. Then, for j equal to $1, \ldots, n$, there exists an element I_j in C such that $J_j \not\leq I_j$. Then $\wedge_{j=1}^n I_j$ is an element of the subset $\{I_1, \ldots, I_n\}$ of C, but is not primal, thereby yielding a contradiction. Using Zorn's lemma, the result now follows. ■

In the sequel, when the lattice of norm-closed ideals of a JB*-triple is considered, the primitive spectrum will always be the primitive subset used. The following result, shows that, for a complete lattice $\mathcal{L}$ possessing a primitive subset $\mathcal{T}$, the topological space $\Omega(\mathcal{L})$ does not depend on the choice of primitive subset $\mathcal{T}$, and may, in fact, be identified with the complete regularisation of the set $\mathrm{Primal}'\mathcal{L}$ of proper primal elements of $\mathcal{L}$.

Theorem 3.17 *Let $\mathcal{L}$ be a complete lattice possessing a primitive subset $\mathcal{T}$, let $\mathrm{Primal}'\mathcal{L}$ be the set of proper primal elements of $\mathcal{L}$, let $\mathrm{Primal}'\mathcal{L}$ and $\mathcal{T}$ be equipped with the relativised lower topology and let $(\gamma_1, \gamma_1\mathrm{Primal}'\mathcal{L})$ and $(\gamma_2, \gamma_2\mathcal{T})$ be the complete regularisations of $\mathrm{Primal}'\mathcal{L}$ and $\mathcal{T}$ respectively. Then, the map $\tilde{\phi}$, defined for I in $\mathrm{Primal}'\mathcal{L}$ by $\tilde{\phi} : \gamma_1(I) \mapsto \gamma_1(I) \cap \mathcal{T}$ is a homeomorphism from $\gamma_1\mathrm{Primal}'\mathcal{L}$ onto $\gamma_2\mathcal{T}$ with inverse defined for P in $\mathcal{T}$ by $\tilde{\psi} : \gamma_2(P) \mapsto \gamma_1(P)$. Let Ω be the subset of $\mathcal{L}$ defined by*

$$\Omega = \{\wedge\gamma_1(I) : I \in \mathrm{Primal}'\mathcal{L}\}.$$

Then, for each element I in $\mathrm{Primal}'\mathcal{L}$, $\wedge\gamma_1(I)$ is the unique element of Ω contained in I. For G in Ω, let $\mathrm{hull}_{\mathrm{Primal}'\mathcal{L}}\, G$ denote the set of proper primal elements of L containing G. Then, the map $\gamma_1(I) \mapsto \wedge\gamma_1(I)$ is a bijection from $\gamma_1\mathrm{Primal}'\mathcal{L}$ onto Ω with inverse $G \mapsto \mathrm{hull}_{\mathrm{Primal}'\mathcal{L}}\, G$, such that

$$\wedge\gamma_1(I) = \wedge(\gamma_1(I) \cap \mathcal{T})$$

The set Ω is a subset of $\mathrm{Primal}'\mathcal{L}$ if and only if Ω and $\mathrm{MinPrimal}\,\mathcal{L}$ coincide as sets.

Proof. Let I be an element of $\mathrm{Primal}'\mathcal{L}$. For any element Q in $\mathrm{hull}_{\mathcal{T}}\, I$, it follows from Proposition 3.12 that $\mathrm{hull}_{\mathcal{T}}\, I$ is a subset of $\gamma_2(Q)$, and, hence, the mapping ϕ defined for I in $\mathrm{Primal}\,\mathcal{L}$ by $\phi(I) = \gamma_2(Q)$ is well defined and onto $\gamma_2\mathcal{T}$. Let g be an element of $C^b(\mathcal{T})$, the continuous bounded functions on $\mathcal{T}$. As in [4], Proposition 3.8, $g\phi$ lies in $C^b(\mathrm{Primal}'\mathcal{L})$, the space of continuous complex-valued bounded functions on $\mathrm{Primal}'\mathcal{L}$. The continuity of ϕ now follows from the fact that the kernels of the continuous complex-valued bounded functions on a completely regular space form a basis for that topology. Let ψ be the restriction of γ_1 to $\mathcal{T}$. Let $\tilde{\phi} : \gamma_1\mathrm{Primal}'\mathcal{L} \mapsto \gamma_2\mathcal{T}$ and $\tilde{\psi} : \gamma_2\mathcal{T} \mapsto \gamma_1\mathrm{Primal}'\mathcal{L}$ be the continuous liftings of ϕ and ψ respectively. For an element P in $\mathcal{T}$,

$$\tilde{\phi}\tilde{\psi}(\gamma_2(P)) = \tilde{\phi}(\gamma_1(P)) = \phi(P) = \gamma_2(P),$$

since P lies in $\mathrm{hull}_{\mathcal{T}}\, P$. Therefore $\tilde{\phi}\tilde{\psi}$ is the identity on $\gamma_2\mathcal{T}$. Let I be an element of $\mathrm{Primal}'\mathcal{L}$ and let Q be an element of $\mathrm{hull}_{\mathcal{T}}\, I$. Then,

$$\tilde{\psi}\tilde{\phi}(\gamma_1(I)) = \tilde{\psi}(\phi(I)) = \tilde{\psi}(\gamma_2(Q)) = \phi(Q) = \gamma_1(Q).$$

By Proposition 3.12, there exists a net in $\mathcal{T}$ $\mathfrak{W}$-convergent to every point of $\mathrm{hull}_{\mathcal{T}}\, I$. By [4], Proposition 3.2, this net converges to Q and I in $\mathrm{Primal}'\mathcal{L}$. Hence $\gamma_1(Q)$ and $\gamma_1(I)$ coincide and $\tilde{\psi}\tilde{\phi}$ is the identity on $\gamma_1\mathrm{Primal}'\mathcal{L}$. The homeomorphism $\tilde{\phi}$ induces an isometric *-isomorphism $f \mapsto f|_{\mathcal{T}}$ from $C^b(\mathrm{Primal}'\mathcal{L})$ onto $C^b(\mathcal{T})$. Therefore, for elements P and Q in $\mathcal{T}$, $\gamma_1(P)$ and $\gamma_1(Q)$ coincide if and only if $\gamma_2(P)$ and $\gamma_2(Q)$ coincide. Thus, for I in $\mathrm{Primal}'\mathcal{L}$ and Q in $\mathrm{hull}_{\mathcal{T}}\, I$,

$$\phi(I) = \gamma_2(Q) = \gamma_1(Q) \cap \mathcal{T} = \gamma_1(I) \cap \mathcal{T}.$$

Let Ω be the set

$$\Omega = \{\wedge\phi(I) : I \in \mathrm{Primal}'\mathcal{L}\}.$$

Then, for I in $\mathrm{Primal}'\mathcal{L}$, the element $\wedge\phi(I)$ of Ω can be written in the form $\wedge\gamma_2(P)$, where P is any element of $\mathrm{hull}_{\mathcal{T}}\, I$. Since $\mathrm{hull}_{\mathcal{T}}\, I$ is a subset of $\gamma_2(P)$, it follows that $\wedge\phi(I)$ is an element of Ω contained in I. Let the element Q in $\mathcal{T}$ be such that $\gamma_2(Q)$ is another element of Ω contained in I and, hence, contained in P. Then, P lies in $\mathrm{hull}_{\mathcal{T}} \wedge\gamma_2(Q)$, and, hence, $\gamma_2(P)$ and $\gamma_2(Q)$ coincide. Thus, every element I in $\mathrm{Primal}'\mathcal{L}$ contains exactly one element $\wedge\phi(I)$ of Ω. Also observe that if J lies in $\mathrm{hull}_{\mathrm{Primal}'\mathcal{L}} \wedge\phi(I)$ then $\wedge\phi(J)$ and $\wedge\phi(I)$ coincide, and are the unique element of Ω contained in J. Conversely, if $\wedge\phi(J)$ and $\wedge\phi(I)$ coincide, then since $\wedge\phi(J)$ is contained in J, so also is $\wedge\phi(I)$. It follows that J is an element of $\mathrm{hull}_{\mathrm{Primal}'\mathcal{L}} \wedge\phi(I)$ if and only if $\wedge\phi(J)$ and $\wedge\phi(I)$ coincide, which occurs if and only if $\gamma_1(I)$ and $\gamma_1(J)$ coincide. Thus

$$\mathrm{hull}_{\mathrm{Primal}'\mathcal{L}} \wedge\phi(I) = \gamma_1(I) \supseteq \gamma_1(I) \cap \mathcal{T} = \phi(I).$$

It follows that

$$\wedge\phi(I) \leq \wedge\gamma_1(I) \leq \wedge\phi(I),$$

and the proof that

$$\wedge\gamma_1(I) = \wedge(\gamma_1(I) \cap \mathcal{T})$$

is complete.

Suppose that Ω is a subset of $\mathrm{Primal}'\mathcal{L}$, and let I be a minimal element of $\mathrm{Primal}'\mathcal{L}$. Then, $\wedge\phi(I)$ is a primal element contained in I, and, hence, equal to I by the minimality of I. Conversely, let I be primal and let J be a primal element dominated by $\wedge\phi(I)$. Then

$$\wedge\phi(J) \leq J \leq \wedge\phi(I) \leq I.$$

Since I contains exactly one element of Ω, J and $\wedge\phi(I)$ coincide. Thus $\wedge\phi(I)$ is a minimal primal ideal and the sets Ω and $\mathrm{MinPrimal}\,\mathcal{L}$ coincide. The converse implication is obvious. ■

Let $\mathcal{L}$ be a complete lattice. For elements J, K of $\mathcal{L}$, J is said to be *way below* K, written $J \ll K$ if and only if, whenever D is a directed subset of $\mathcal{L}$ such that $K \leq \vee D$, there exists an element I in D such that $J \leq I$. For elements I, J, K and L of $\mathcal{L}$,

(i) $J \ll K$ implies $J \leq K$;

(ii) $I \leq J \ll K \leq L$ implies $I \ll L$;

(iii) $J \ll L$ and $K \ll L$ implies $J \vee K \ll L$;

(iv) $0 \ll J$.

For a subset S of $\mathcal{L}$, write

$$\Downarrow S = \{K \in \mathcal{L} : K \ll J \text{ for some } J \in S\}, \qquad \Downarrow J = \Downarrow \{J\},$$
$$\Uparrow S = \{K \in \mathcal{L} : K \gg J \text{ for some } J \in S\}, \qquad \Uparrow J = \Uparrow \{J\}.$$

A complete lattice $\mathcal{L}$ is said to be *continuous* if it satisfies the *axiom of approxi-*

mation that, for each element J in $\mathcal{L}$,

$$J = \vee\{I \in \mathcal{L} : I \ll J\}.$$

It follows from [53], Theorem 4.9 that a continuous lattice is a compact Hausdorff space in the Lawson topology.

Proposition 3.18 *Let $\mathcal{L}$ be a continuous lattice possessing a primitive subset $\mathcal{T}$, let* 1 *be the maximal element of $\mathcal{L}$, let* $\mathrm{Primal}\,\mathcal{L}$ *be the set of primal ideals, let* $\mathrm{Primal}'\,\mathcal{L}$ *be the set of proper primal ideals, let $\mathfrak{W}$ be the lower topology and let $\mathfrak{L}$ be the Lawson topology. Then,* $\mathrm{Primal}\,\mathcal{L}$ *is $\mathfrak{L}$-compact and the following conditions are equivalent:*

(i) $\mathrm{Primal}'\,\mathcal{L}$ *is $\mathfrak{L}$-compact;*

(ii) $\mathrm{Primal}'\,\mathcal{L}$ *is $\mathfrak{W}$-compact;*

(iii) $\mathcal{T}$ *is $\mathfrak{W}$-compact;*

(iv) 1 *is not an element of* $\overline{\mathrm{Primal}'\,\mathcal{L}}^{\mathfrak{L}}$*;*

(v) $\{1\}$ *is $\mathfrak{L}$-open in* $\mathrm{Primal}\,\mathcal{L}$*.*

Proof. By Theorem 3.11, $\mathrm{Primal}\,\mathcal{L}$ is $\mathfrak{W}$-closed, and, hence, $\mathfrak{L}$-closed. Since $\mathcal{L}$ is continuous, $\mathcal{L}$ is $\mathfrak{L}$-compact and Hausdorff and hence $\mathrm{Primal}\,\mathcal{L}$ is $\mathfrak{L}$-compact.

The implications (i)⇒(ii), (i)⇔(iv)⇔(v) are immediate. That (ii)⇔(iii) can be seen using the argument of [53], Theorem 7.4. To show that (ii)⇒(iv), let $\mathfrak{S}$ be the Scott topology of $\mathcal{L}$. Suppose that (ii) holds and assume that 1 is an element of $\overline{\mathrm{Primal}'\,\mathcal{L}}^{\mathfrak{L}}$, and hence $\overline{\mathrm{Primal}'\,\mathcal{L}}^{\mathfrak{S}}$ Then, by [53], Theorem 5.2(a), 1 is an element of $\mathrm{Primal}'\,\mathcal{L}$, therefore yielding a contradiction. ■

Proposition 3.19 *Let X be a topological space, and let $\mathcal{O}(X)$ be the complete lattice of open sets. Then, the following results hold.*

(i) *Let U, V be open sets for which there exists a compact set K such that*

$$U \subseteq K \subseteq V.$$

Then $U \ll V$.

(ii) *Let X be locally compact and let U and V be open sets such that $U \ll V$. Then, there exists a compact set K such that*

$$U \subseteq K \subseteq V$$

and the complete lattice $\mathcal{O}(X)$ of open sets is continuous.

Theorem 3.20 *Let $\mathcal{L}$ be a complete lattice possessing a primitive subset $\mathcal{T}$ which is locally compact in the lower topology. For every compact subset C of $\mathcal{T}$ and every finite subset $\mathcal{F}$ of $\mathcal{L}$, define the subset $\mathcal{U}(C, \mathcal{F})$ of $\mathcal{L}$ by*

$$\mathcal{U}(C, \mathcal{F}) = \{J \in \mathcal{L} : J \not\leq K \, \forall K \in C, K \not\leq J \, \forall K \in \mathcal{F}\}.$$

Then:

(i) *$\mathcal{L}$ is a continuous lattice;*

(ii) *the collection of sets of the form $\mathcal{U}(\emptyset, \mathcal{F})$ forms a basis for the lower topology;*

(iii) *the collection of sets of the form $\mathcal{U}(C, \emptyset)$ forms a basis for the Scott topology;*

(iv) *the collection of sets of the form $\mathcal{U}(C, \mathcal{F})$ forms a basis for the Lawson topology.*

Proof. The map $J \mapsto \mathcal{T} \setminus \text{hull}_{\mathcal{T}} J$ is an order isomorphism onto the lower topology of $\mathcal{T}$. It follows from Proposition 3.19 that (i) holds. Part (ii) is immediate from the definition. Parts (iii) and (iv) can be found in [47]. ∎

When $\mathcal{T}$ is as in Theorem 3.20, $\overline{\mathcal{T}}^{\mathfrak{L}}$, the closure of $\mathcal{T}$ in the Lawson topology is said to be the *Hausdorff compactification* of $\mathcal{T}$ [43]. The following lemma, proved in

[43], shows that $\overline{\mathcal{T}}^{\mathfrak{L}}$ has a particularly simple form when $\mathcal{T}$ itself is Hausdorff in the lower topology.

Lemma 3.21 *Let $\mathcal{L}$ be a complete lattice possessing a primitive subset $\mathcal{T}$ which is locally compact and Hausdorff in the lower topology, and let $\overline{\mathcal{T}}^{\mathfrak{L}}$ be the closure of $\mathcal{T}$ in the Lawson topology. When T is compact*

$$\overline{\mathcal{T}}^{\mathfrak{L}} = \mathcal{T},$$

and, otherwise,

$$\overline{T}^{\mathfrak{L}} = T \cup \{1\}.$$

3.3 Banach space spectra

Having established all of the lattice theoretic results that will be required in this thesis in Section 3.2, some of the tools used to explore the structure of a Banach space are described in this section, starting with pure functionals and the centralizer.

Let A be a Banach space with dual A^* and let $\partial_e A_1^*$ be the set of extreme points of A_1^*, the unit ball of A^*. The elements of $\partial_e A_1^*$ are said to be the *pure functionals* of A.

Proposition 3.22 *Let A be a complex Banach space with dual A^* and let $\partial_e A_1^*$ be the set of pure functionals of A. For an operator T in the Banach algebra $B(A)$ of bounded linear operators on A, let T^* be the dual operator on A^*. Let $Z(A)$ be the set of elements T in $B(A)$ for which there exists an operator $T^\dagger$ in $B(A)$ such that, for every element x in $\partial_e A_1^*$, there exists $\lambda_{T,x}$ in $\mathbb{C}$, such that,*

$$T^* x = \lambda_{T,x} x$$
$$(T^\dagger)^* x = \overline{\lambda_{T,x}} x.$$

Then:

(i) *for each element* T *in* $Z(A)$, *and* x *in* $\partial_e A_1^*$, *the scalar* $\lambda_{T,x}$ *is unique and the function* $\check{T} : x \mapsto \lambda_{T,x}$ *is* w^**-continuous on* $\partial_e A_1^*$ *and bounded by* $\|T\|$*;*

(ii) *for each element* T *in* $Z(A)$, $T^\dagger$ *is unique;*

(iii) $Z(A)$ *is a commutative closed subalgebra of* $B(A)$ *and the mapping* $T \mapsto T^\dagger$ *is an involution on* $Z(A)$ *with respect to which* $Z(A)$ *is a commutative unital C*-algebra;*

(iv) *for each element* x *in* $\partial_e A_1^*$, *the function* $\check{x} : T \mapsto \lambda_{T,x}$ *is a character of* $Z(A)$.

For a Banach space A, the C*-algebra $Z(A)$ defined in Proposition 3.22 is said to be the *centralizer* of A. The Boolean algebra of projections of $Z(A)$ coincides with the Boolean algebra of M-projections of A, and, when A is a dual space, $Z(A)$ is the Banach *-subalgebra of $B(A)$ generated by the complete Boolean algebra of M-projections. Thus, the centralizer is a convenient tool for studying the M-summands of a Banach space. In Chapter 6, the centralizer will be used to study a class of norm-closed ideals in a JBW*-triple, and, in Chapter 7, it will be used to characterise the pure state space of a continuous JBW*-triple.

The centralizer is said to be trivial if it consists of scalar multiples of the identity. This occurs whenever A has no non-trivial M-ideals. The partial converse is that if $Z(A)$ is trivial then A has no non-trivial M-summands.

Let A be a Banach space. An M-ideal of A is said to be respectively *prime*, *primal* or *maximal* if it is prime, primal or maximal as an element of $\mathcal{ZI}_{\mathrm{n}}(A)$, the lattice of M-ideals. The set of prime ideals of A is denoted by $\mathrm{Prime}\, A$ and the set of primal ideals by $\mathrm{Primal}\, A$. However, the subset of $\mathcal{ZI}_{\mathrm{n}}(A)$ which will be the most useful for studying the structure of A is the primitive spectrum.

Let L be a norm-closed subspace of A and define the *central kernel* $k(L)$ of L and the *norm central kernel* $k_n(L)$ of L by

$$k(L) = \bigvee \{I \in \mathcal{ZI}(A) : I \subseteq L\}$$
$$k_n(L) = \bigvee \{I \in \mathcal{ZI}_{\mathrm{n}}(A) : I \subseteq L\}.$$

Since L is a norm-closed subspace, $k(L)$ is the largest M-summand contained in L and $k_n(L)$ is the largest M-ideal contained in L.

Let A be a Banach space with dual space A^*, let A_1^* be the closed unit ball in A^* and let $\partial_e A_1^*$ be the set of extreme points of A_1^*. For each element x in $\partial_e A_1^*$, let L_x be the smallest L-summand of A^* containing x and let K_x be the smallest w*-closed L-summand of A^* containing x. It is clear that L_x is a minimal L-summand of A^*. The *spectrum* of A is defined to be the set $\{L_x : x \in \partial_e A_1^*\}$ and the *primitive spectrum* of A ([28]) is defined to be the set $\{K_x : x \in \partial_e A_1^*\}$. For x in $\partial_e A_1^*$, define the *central kernel* of x in A^{**} to be the central kernel $k(\ker^* x)$ of the kernel $\ker^* x$ of x in A^{**} and define the *norm central kernel* of x to be the norm central kernel $k_n(\ker_* x)$ of the kernel $\ker_* x$ of x in A. Then the mappings $L_x \mapsto L_x^\circ$ and $K_x \mapsto (K_x)_\circ$ are bijections from the spectrum and the primitive spectrum onto the sets

$$\operatorname{Spec} A = \{k(\ker^* x) : x \in \partial_e A_1^*\}$$
$$\operatorname{Prim} A = \{k_n(\ker_* x) : x \in \partial_e A_1^*\}$$

respectively. In the sequel, by slight abuse of terminology, $\operatorname{Spec} A$ will be said to be the *spectrum* of A and $\operatorname{Prim} A$ will be said to be the *primitive spectrum* of A.

It follows from equation (3.1.1) that

$$\operatorname{Prim} A \subseteq \operatorname{Prime} A \subseteq \operatorname{Primal} A.$$

Given any ideal J, the hull of J is defined by

$$\operatorname{hull} J = \operatorname{hull}_{\operatorname{Prim} A} J = \{P \in \operatorname{Prim} A : J \subseteq P\}.$$

Proposition 3.23 below shows that $\operatorname{Prim} A$ is order generating. Hence, by Theorem 3.11, the closures of $\operatorname{Prim} A$ and $\operatorname{Primal} A$ in the lower topologies coincide with $\operatorname{Primal} A$. As in Section 3.2, the sets $\operatorname{hull} J$ for J in $\mathcal{ZI}_{\mathrm{n}}(A)$ are the closed sets of the relative lower topology on $\operatorname{Prim} A$. This topology is known as the *Jacobson topology* and is denoted by $\mathfrak{J}$.

Proposition 3.23 *Let A be a Banach space with primitive spectrum* $\mathrm{Prim}\,A$ *and let* $\mathfrak{J}$ *be the Jacobson topology on* $\mathrm{Prim}\,A$. *Then, the following results hold.*

(i) *For each M-ideal J in A, with hull,* $\mathrm{hull}\,J$,

$$J = \bigcap\{P : P \in \mathrm{hull}\,J\}.$$

(ii) *For each subset S of* $\mathrm{Prim}\,A$, *let* $\overline{S}^{\mathfrak{J}}$ *be the closure of S in* $\mathrm{Prim}\,A$ *in the Jacobson topology and let $k_n(\cap S)$ be the norm central kernel of $\cap S$. Then*

$$\overline{S}^{\mathfrak{J}} = \mathrm{hull}\,k_n(\cap S).$$

Proof. Let K be a w*-closed L-summand of A^*. Since K is w*-closed, K_1 is a w*-compact convex subset of A^*. Let $\partial_e K_1$ be the set of extreme points of K_1. By the Krein-Milman theorem,

$$K_1 \subseteq \overline{\mathrm{conv}\,\partial_e K_1}^{w*} \subseteq \overline{\mathrm{lin}\{K_x : x \in \partial_e K_1\}}^{w*} \subseteq \bigvee\{K_x : x \in \partial_e K_1\} \subseteq K.$$

Thus

$$K = \overline{\mathrm{lin}\{K_x : x \in \partial_e K_1\}}^{w*} = \bigvee\{K_x : x \in \partial_e K_1\}.$$

Equivalently, for an M-ideal J of A,

$$J = \bigwedge\{J_x : x \in \partial_e J_1^\circ\} = \bigwedge\{J_x : x \in \partial_e A_1^* \cap J^\circ\} = \bigwedge\{P : P \in \mathrm{hull}\,J\}.$$

Thus $\mathrm{Prim}\,A$ order generates $\mathcal{ZI}_{\mathrm{n}}(A)$ and the result follows from Proposition 3.10. ∎

Lemma 3.24 *Let A be a JB*-triple, let $\partial_e A_1^*$ be the set of pure functionals of A, let* $\mathrm{Spec}\,A$ *be the spectrum of A and let* $\mathrm{Prim}\,A$ *be the primitive spectrum of A, equipped with the Jacobson topology $\mathfrak{J}$. For x in $\partial_e A_1^*$, let J^x be the central kernel of x in A^{**}, let J_x be the norm central kernel of x in A and define surjections $\Theta : \partial_e A_1^* \mapsto$* $\mathrm{Spec}\,A$,

$\Phi : \operatorname{Spec} A \mapsto \operatorname{Prim} A$ *and* $\Psi : \partial_e A_1^* \mapsto \operatorname{Prim} A$ *by*

$$\begin{aligned} \Theta(x) &= J^x \\ \Phi(J^x) &= J_x \\ \Psi(x) &= \Phi \circ \Theta(x) = J_x. \end{aligned}$$

Let $\mathfrak{J}'$ *be the quotient topology induced on* Spec A *from* Prim A *by* Φ *and let* $\mathfrak{J}''$ *be the quotient topology induced on* $\partial_e A_1^*$ *from* Prim A *by* Ψ*. Let* J *be an M-ideal of* A*, with hull* hull J*. Then*

$$\Psi^{-1}(\operatorname{hull} J) = J^\circ \cap \partial_e A_1^*,$$

$\mathfrak{J}''$ *is weaker than the relative w*-topology on* $\partial_e A_1^*$, Ψ *is w* to* $\mathfrak{J}$ *continuous and* Θ *is w* to* $\mathfrak{J}'$ *continuous.*

Proof. For each element x of $\partial_e A_1^*$, let K_x be the smallest w*-closed L-summand of A^* containing x and let L_x be the smallest norm-closed L-summand of A^* containing x. Let x and y be elements of $\partial_e A_1^*$ such that L_x and L_y coincide. Then L_y and hence K_y are subsets of K_x, and by symmetry, K_x and K_y coincide. Thus Φ is well defined. The subsets of $\partial_e A_1^*$ closed in the structure topology are of the form

$$\begin{aligned} \Psi^{-1}(\operatorname{hull} J) &= \{x \in \partial_e A_1^* : J \subseteq J_x\} \\ &= J^\circ \cap \partial_e A_1^*. \end{aligned}$$

Since J° is w*-closed, it follows that the structure topology is weaker than the w*-topology on $\partial_e A_1^*$. ■

The topologies $\mathfrak{J}'$ and $\mathfrak{J}''$ of Lemma 3.24 are said to be the *structure topologies* of their respective spaces.

Lemma 3.25 *Let* A *be a JB*-triple and adopt the notation of Lemma 3.24. Let* S *be a subset of* Spec A *and let* x *be an element of* $\partial_e A_1^*$*. Then* J^x *lies in* $\overline{S}^{\mathfrak{J}'}$ *if and only if* $\cap_{J \in S} \Phi(J)$ *is a subset of* J_x*.*

Proof. The result follows from the equation

$$\overline{S} = \Phi^{-1}(\overline{\Phi(S)}^{\mathfrak{J}}).$$

■

An early indication of the relevance of the primitive spectrum to questions concerning Banach space structure is given by the following well known theorem, often referred to as the Dauns-Hofmann theorem. It was proved for C*-algebras in [22], real Banach spaces in ([2], Section 4) and extended to complex Banach spaces in ([13], Chapter 3).

Theorem 3.26 *Let A be a Banach space with primitive spectrum* $\mathrm{Prim}\, A$ *equipped with the Jacobson topology $\mathfrak{J}$ and let $\partial_e A_1^*$ be the set of pure functionals of A, equipped with the structure topology $\mathfrak{J}''$. Then, for each element T of the centralizer $Z(A)$ of A, there exists a function $\check{T}$ of the space $C^b(\partial_e A_1^*)$ of complex-valued bounded $\mathfrak{J}''$-continuous functions on $\partial_e A_1^*$, such that for all elements x of $\partial_e A_1^*$,*

$$T^* x = \check{T}(x)x,$$

and a function f_T in the space $C^b(\mathrm{Prim}\, A)$ of complex-valued bounded $\mathfrak{J}$-continuous functions such that for all elements x of $\partial_e A_1^*$,

$$f_T(J_x) = \check{T}(x),$$

*where J_x is the norm central kernel $k_n(\ker x)$ of x. The mapping $T \mapsto \check{T}$ is a *-isomorphism from $Z(A)$ onto $C^b(\partial_e A_1^*)$ and the mapping $\check{T} \mapsto f_T$ is a *-isomorphism from $C^b(\partial_e A_1^*)$ onto $C^b(\mathrm{Prim}\, A)$.*

3.4 Representations

Representations are important in the study of JB*-triples as a means of accessing the more complete theory of JBW*-triples, and as a way of relating general JB*-triples to more concrete examples. In this section representation theory for Banach spaces is introduced.

Let A and M be dual Banach spaces and choose preduals A_* and M_*. It will be understood that the w*-topologies on A and M are those induced by the chosen preduals. Let $\pi : A \mapsto M$ be a w*-continuous linear map such that $\ker \pi$ is an M-summand of A and $\pi(A)$ is a w*-dense in M. The *central support* of π, written $\operatorname{spt} \pi$, is defined to be $(\ker \pi)^\perp$, the M-orthogonal complement of $\ker \pi$. If π is an isometry on $\operatorname{spt} \pi$ then (π, M) is said to be a *w*-representation* of A.

Let A be an arbitrary Banach space, let M be a dual Banach space with chosen predual M_* and let $\pi : A \mapsto M$ be a bounded linear mapping. Then (π, M) is said to be a *representation* of A if the unique w*-continuous extension $\tilde{\pi} : A^{**} \mapsto M$ is a w*-representation of A^{**} (where A^* is the chosen predual of A^{**}). The *support* of π, denoted by $\operatorname{spt} \pi$, is defined to be equal to $\operatorname{spt} \tilde{\pi}$.

Lemma 3.27 *Let A be a dual Banach space and let (π, M) be a w*-representation of A. Then $\pi(A)$ is equal to M.*

Proof. Since $\operatorname{spt} \pi$ is an M-summand, it is w*-closed and its unit ball $\operatorname{spt} \pi \cap A_1$ is w*-compact. Since π is w*-continuous and an isometry on its support, $\pi(A) \cap M_1$ coincides with $\pi(\operatorname{spt} \pi_1)$ and is w*-compact. It follows by the Krein-Šmulian theorem that $\pi(A)$ is w*-closed. ■

Corollary 3.28 *Let A be an arbitrary Banach space and let (π, M) be a representation of A with extension $\tilde{\pi}$ to the bidual A^{**} of A. Then*

$$M = \tilde{\pi}(A^{**}) = \overline{\pi(A)}^{w*}.$$

Proof. Since A is w*-dense in A^{**} and $\tilde{\pi}$ is w*-continuous, $\tilde{\pi}(A)$ is w*-dense in $\tilde{\pi}(A^{**})$. ■

Two w*-representations (π_1, M_1) and (π_2, M_2) of a dual Banach space A are said to be *quasi-equivalent* if $\operatorname{spt}\pi_1$ and $\operatorname{spt}\pi_2$ coincide. Clearly this is equivalent to the existence of a linear isometry ψ from $\pi_1(A)$ onto $\pi_2(A)$ such that $\psi \circ \pi_1$ agrees with π_2. The w*-representations are said to be *disjoint* if $\operatorname{spt}\pi_1$ and $\operatorname{spt}\pi_2$ have zero intersection. Two representations of an arbitrary Banach space A are said to be quasi-equivalent or disjoint if their extensions to A^{**} are quasi-equivalent or disjoint respectively. A representation is said to be *factorial* if its support is a factor.

Let J be an M-summand of a dual space A and let P_J be the M-projection with range J. Then J is w*-closed and therefore a dual space. Hence (P_J, J) is a w*-representation of A. Clearly every representation (π, M) of A is quasi-equivalent to $(P_{\operatorname{spt}\pi}, \operatorname{spt}\pi)$. Thus the representation theory of a dual space reduces to its M-summand theory, and the representation theory of an arbitrary space reduces to the M-summand theory of its second dual.

Let A be a Banach space and let $\iota : A \mapsto A^{**}$ be the natural inclusion. For each element x in A^*, recall that L_x is the smallest L-summand of A^* containing x. Associate with each element x in A^* a representation (π_x, M_x) where M_x is the dual Banach space L_x^* and π_x is the natural representation of A onto a w*-dense subset of L_x^*. Then,

$$\operatorname{Spec} A = \{\ker \widetilde{\pi}_x : x \in \partial_e A_1^*\},$$

and

$$\operatorname{Prim} A = \{\ker \pi_x : x \in \partial_e A_1^*\}.$$

Chapter 4

JB*-triple ideals

This chapter continues the discussion of M-structure begun in Chapter 3, but specialises to the JB*-triple case, where the extra structure leads to a richer theory.

In Section 4.1 the M-structure of a JB*-triple A is connected to its algebraic structure and some important consequences are recalled. In Section 4.2 an algebraic expression for the norm central kernel of a norm-closed subspace of a JB*-triple is found. In Section 4.3, a different approach is adopted to give an alternative description of the norm central kernel of a functional on a JB*-triple. In Sections 4.4 and 4.5 some additional topological results for the spectrum and primitive spectrum respectively are obtained. In Section 4.6 an alternative description of the lower, Scott and Lawson topologies, introduced in the lattice setting in Section 3.2, is given. Finally, in Section 4.7, more information is obtained about the complete regularisation of the primitive spectrum and the Dauns-Hofmann theorem in the JB*-triple case.

4.1 JB*-triple ideals

In this section some well known results from the rich theory connecting the M-structure of a JB*-triple to its algebraic structure are recalled.

The significance of M-structure in JB*-triples and JBW*-triples is explained by the following result, which follows from [18], Proposition 1.3, [9], Theorem 3.2 and Proposition 3.5.

Theorem 4.1 *Let A be a JB*-triple and let J be a norm-closed subspace of A. Then the following are equivalent:*

(i) *J is an ideal of A;*

(ii) $\{A\,A\,J\} \subseteq J$;

(iii) $\{A\,J\,A\} \subseteq J$;

(iv) $\{A\,J\,J\} \subseteq J$;

(v) *J is an M-ideal of A.*

When A is a JBW-triple and J is a w*-closed subspace, conditions* (i) *to* (v) *are equivalent to the condition that J is an M-summand of A.*

Let J be an element of the complete Boolean algebra $\mathcal{ZI}(A)$ of w*-closed ideals of a JBW*-triple A. Then, an immediate consequence of [34] is that the algebraic annihilator of J as defined in Section 2.1.3 coincides with the complementary M-summand of J, justifying the use of the notation $J^{\perp}$ for both concepts. Let A be a JB*-triple with complete lattice of norm-closed ideals $\mathcal{ZI}_{\mathrm{n}}(A)$. Another fundamental consequence of Theorem 4.1 is that the intersection of any family $\{J_\lambda : \lambda \in \Lambda\}$ in $\mathcal{ZI}_{\mathrm{n}}(A)$, is again a norm-closed ideal and

$$\bigwedge\{J_\lambda : \lambda \in \Lambda\} = \bigcap\{J_\lambda : \lambda \in \Lambda\}.$$

It follows that, for a norm-closed subspace L of A, the *norm central hull* $c_n(L)$ of L, defined by

$$c_n(L) = \bigwedge\{I \in \mathcal{ZI}_{\mathrm{n}}(A) : L \subseteq I\}$$

is the smallest M-ideal of A containing L.

When A is a JBW*-triple and L a w*-closed subspace of A, the *central hull* $c(L)$ of L is defined by

$$c(L) = \bigwedge\{I \in \mathcal{ZI}(A) : L \subseteq I\}.$$

Thus the central hull is the smallest w*-closed ideal containing L and the central kernel is the largest w*-closed ideal contained in L. Central hulls and central kernels in JB*-triples have been the subject of intensive study, see, for example, [20], [36], [40], [41]. Theorem 4.2 can be found in [40], Lemma 3.12.

Theorem 4.2 *Let A be a JBW*-triple and let J be a w*-closed inner ideal of A. Then*

$$c(J)^{\perp} = k(J^{\perp}).$$

It is now possible to give an algebraic characterisation of the representations of a JB*-triple.

Proposition 4.3 *Let A be a JB*-triple, let M be a JBW*-triple and let (π, M) be a representation of A with extension $(\tilde{\pi}, M)$ to A^{**}. Then, π and $\tilde{\pi}$ are homomorphisms.*

Proof. By definition, $\tilde{\pi}$ is an isometry on $\operatorname{spt}\tilde{\pi}$, the support of $\tilde{\pi}$. By Theorem 2.13, $\tilde{\pi}$ is an isometric isomorphism from $\operatorname{spt}\tilde{\pi}$ onto M. By Theorem 4.1, $\tilde{\pi}$ is a homomorphism on A^{**}. The result follows. ■

By a slight abuse of terminology, if B is a JBW*-triple containing M as a JBW*-subtriple, the pair (π, B) will be said to be a *representation* of A.

The bounded linear operators on a JB*-triple which lie in the centralizer can also be given an algebraic characterisation. Let A be a Jordan *-triple and recall from Chapter 2, that for a and b in A, a linear operator $D(a)$ is defined by

$$D(a)b = \{a\,a\,b\}.$$

The *centroid* of A is the algebra of linear operators $T : A \mapsto A$ which commute with all operators of the form $D(a)$ for a in A. When A is also a Banach space, the *continuous centroid* is defined to be the algebra of bounded elements of the centroid. The continuous centroid of a JB*-triple A coincides with the centralizer $Z(A)$ of A, and the involution $T \mapsto T^{\dagger}$ on $Z(A)$, defined in Proposition 3.22, satisfies

$$T\{a\,b\,c\} = \{a\,T^{\dagger}b\,c\}$$

for all elements a, b and c in A [24]. Proposition 4.4 ([24], Proposition 3.5) gives a more familiar algebraic interpretation of the centralizer in the case of a unital JB*-algebra.

Proposition 4.4 *Let A be a JB*-algebra with unit 1 and centralizer $Z(A)$. Then the map $T \mapsto T1$ is an isometric *-isomorphism from $Z(A)$ onto the centre of A.*

Theorem 4.5 was proved in [36]. An equivalent result for JB*-triples possessing a complete tripotent will be given in Corollary 6.5.

Theorem 4.5 *Let A be a JBW*-triple with centralizer $Z(A)$ and let J be a w*-closed inner ideal in A with centralizer $Z(J)$. Let $P_{c(J)}$ be the M-projection of A onto the central hull $c(J)$ of J. Then $Z(c(J))$, the centralizer of $c(J)$ coincides with the w*-closed ideal $P_{c(J)}Z(A)$ of $Z(A)$ and the map $T \mapsto T|_J$ is a *-isomorphism from $Z(c(J))$ onto $Z(J)$.*

One immediate consequence of Theorem 4.5 is that the set of w*-closed ideals of a w*-closed inner ideal in a JBW*-triple may be identified with the set of w*-closed ideals of the central hull. Proposition 4.6 ([20], Lemma 2.4, Proposition 2.5) shows that this result is partially preserved for norm-closed inner ideals in general JB*-triples.

Proposition 4.6 *Let A be a JB*-triple and let I be a norm-closed inner ideal of A. Then, the following results hold.*

(i) *If J is an M-ideal of A, $I \cap J$ is an M-ideal of I and*

$$I \cap J = \{I\,J\,I\}$$
$$c_n(I \cap J) = c_n(I) \cap J.$$

(ii) *If J is an M-ideal of I then*

$$J = c_n(J) \cap I.$$

Theorem 4.7 is a powerful tool in the study of w*-closed inner ideals in JBW*-triples. It is an amalgamation of results from [32], Section 5.

Theorem 4.7 *Let A be a JBW*-triple with predual A_* and let J be a w*-closed inner ideal in A. Then there exists a unique structural projection P_J on A such that*

$$J = P_J A$$

and J is the dual of the subspace

$$J_\# = (P_J)_* A_* = \{x \in A_* : \|x\| = \|x|_J\|\} = \{x \in A_* : e(x) \in J\}$$

of A_*.

It is important to note that the isometric isomorphism of Theorem 4.8 ([9], Proposition 3.4) is not the usual inclusion of a Banach space in its bidual.

Theorem 4.8 *Let A be a JBW*-triple with predual A_*, dual A^* and bidual A^{**}. Then A_* is an L-summand of A^* and A is isometrically isomorphic to a w*-closed ideal of the JBW*-triple A^{**}.*

4.2 Norm central kernels

The main result of this section, Theorem 4.15, is an algebraic characterisation of the norm central kernel $k_n(L)$ of a norm-closed subspace L of a JB*-triple A. The proof will proceed by investigating various subsets of L. Theorem 4.15 is a good example of a result which has a very simple proof in the special case of C*-algebras, but appears only to admit a highly technical proof in the general JB*-triple case. Theorem 4.15 will be used in Chapter 6, but may also be of independent interest.

Throughout this section, for elements a and b in the Jordan *-triple A, recall that $D(a, b)$ and $Q(a, b)$ denote the linear operators on A defined, for each element c in A, by

$$D(a, b)c = \{a\,b\,c\}, \qquad Q(a, b)c = \{a\,c\,b\}$$

and that $D(a)$ and $Q(a)$ denote the operators $D(a, a)$ and $Q(a, a)$ respectively.

Proposition 4.9 *Let A be a JB*-triple, let L be a norm-closed subspace of A and let K_2 be the set*

$$K_2 = \{a \in A : D(b)a \in L \, \forall \, b \in A\}.$$

Then

$$K_2 = \{a \in A : D(b, c)a \in L \, \forall \, b, c \in A\}.$$

and K_2 is a norm-closed inner ideal of A contained in L.

Proof. It is clear that K_2 is a norm-closed subspace of A. By polarisation,

$$K_2 = \{a \in A : D(b, c)a \in L \, \forall \, b, c \in A\}.$$

Let a be an element of K_2 and let b and c be elements of A. Then, by Theorem 2.2,

$$\begin{aligned} D(b)\{a\,c\,a\} &= 2\{\{b\,b\,a\}\,c\,a\} - \{a\,\{b\,b\,c\}\,a\} \\ &= 2D(\{b\,b\,a\}, c)a - D(a, \{b\,b\,c\})a \end{aligned}$$

and $D(b)\{a\,c\,a\}$ lies in L. It follows that $\{a\,c\,a\}$ lies in K_2 and K_2 is a norm-closed inner ideal of A. By the functional calculus (see [24], Proposition 1.4) there exists a sequence (u_n) in $A(a)$, the JB*-subtriple of A generated by a, such that $D(u_n)a$ converges to a. It follows that K_2 is a subset of L. ■

Let A be a JB*-triple, let L be a norm-closed subspace and define a subset K_4 of A by

$$K_4 = \{a \in A : D(a)b \in L \, \forall \, b \in A\}.$$

It is not obvious *a priori* that K_4 is even a subspace of A. To prove that K_4 is in fact a norm-closed inner ideal of A, it must first be established that for a in K_4 and b in A, $\{a\,b\,a\}$ lies in L. This will be proved by proving an equivalent result for powers of a and then using the functional calculus. Lemma 4.10 begins the process.

Lemma 4.10 *Let A be a Jordan *-triple, let L be a subspace of A, let a in A and m in $\mathbb{N}$ be such that $D(a, a^m)b$ and $D(a, a)b$ lie in L for all b in A. Then, for n in $\mathbb{N}_0$, and all b in A,*

$$D(a, a^{4n+m})b \in L, \qquad Q(a, a^{4n+m+2})b \in L.$$

Proof. In the case when n is equal to zero, by hypothesis, $D(a, a^m)b$ lies in L for all b in A. Also,

$$\begin{aligned} Q(a, a^{m+2})b &= \{a\, b\, \{a\, a^m\, a\}\} \\ &= \{\{a\, b\, a\}\, a^m\, a\} \\ &= D(a, a^m)\{a\, b\, a\} \end{aligned}$$

which lies in L by hypothesis.

In the case when n is equal to k, assume that for all b in A,

$$D(a, a^{4k+m})b \in L, \qquad Q(a, a^{4k+m+2})b \in L.$$

By [62], JP9,

$$2D(a, a)D(a^{4k+m+2}, a)b = Q(a, a^{4k+m+2})Q(a)b + D(a, a^{4(k+1)+m})b$$

Therefore $D(a, a^{4(k+1)+m})b$ lies in L. Also,

$$\begin{aligned} Q(a, a^{4(k+1)+m+2})b &= \{a\, b\, \{a\, a^{4(k+1)+m}\, a\}\} \\ &= \{\{a\, b\, a\}\, a^{4(k+1)+m}\, a\} \\ &= D(a, a^{4(k+1)+m})\{a\, b\, a\}. \end{aligned}$$

The result now follows by induction. ■

Lemma 4.11 will be used to extend Lemma 4.10 to all powers, by applying the functional calculus.

Lemma 4.11 *Let A be a JB*-triple, let L be a norm-closed subspace of A and let K_4 be the set*

$$K_4 = \{a \in A : D(a)b \in L \,\forall\, b \in A\}.$$

Then for a in K_4 and b in A, $D(a, a^3)b$ lies in L.

Proof. Let $A(a)$ be the JB*-triple generated by a in A. Then, by Theorem 2.12 there exists a locally compact subset $\sigma_A(a)$ of $(0, \infty)$ and an isometric triple isomorphism from $A(a)$ onto $C_0(\sigma_A(a))$ such that for j in $\mathbb{N}$, a^{2j+1} is mapped to ι^{2j+1}. Let B be the norm-closed *-subalgebra

$$B = \overline{\text{lin}\{\iota^{4j} : j \in \mathbb{N}\}}^n.$$

of $C_0(\sigma_A(a))$. Then, ι^4 is non-zero on $\sigma_A(a)$ and separates points in $\sigma_A(a)$. By the locally compact Stone-Weierstrass Theorem, B coincides with $C_0(\sigma_A(a))$. Given $\varepsilon > 0$, there exists an element of B of the form $\Sigma_{j=1}^n \alpha_j \iota^{4j}$ such that

$$\|\Sigma_{j=1}^n \alpha_j \iota^{4j} - \iota^2\|_\infty < \varepsilon / \|a\|.$$

Therefore,

$$\|\Sigma_{j=1}^n \alpha_j a^{4j+1} - a^3\| \leq \|\iota\|_\infty \|\Sigma_{j=1}^n \alpha_j \iota^{4j} - \iota^2\|_\infty < \varepsilon.$$

Therefore a^3 lies in the set

$$\overline{\text{lin}\{\iota^{4j+1} : j \in \mathbb{N}\}}^n.$$

By Lemma 4.10, $D(a, a^3)b$ lies in L. ∎

Corollary 4.12 *Let A be a JB*-triple, let L be a norm-closed subspace of A and let K_4 be the set*

$$K_4 = \{a \in A : D(a)b \in L \,\forall\, b \in A\}.$$

Then for a in K_4, b in A, and n in $\mathbb{N}_0$, $D(a, a^{2n+1})b$, $Q(a, a^{2n+3})b$ and $\{a\, b\, a\}$ lie in L.

Proof. By Lemma 4.10 and Lemma 4.11, $D(a, a^{2n+1})b$, $Q(a, a^{2n+3})b$ and lie in L. For j in $\mathbb{N}$,

$$(a^3)^{2j+1} = a^{6j+3} = a^{2(3j)+3}$$

Therefore, $Q(a, c)b$ lies in L for all c in A_{a^3}, the JB*-subtriple of A generated by a^3. By [18], Lemma 1.2, a lies in A_{a^3} and the result follows. ∎

Enough technical results have now been established for the investigation of the space K_4 to proceed.

Proposition 4.13 *Let A be a JB*-triple, let L be a norm-closed subspace of A and define subsets K_2 and K_4 of A by*

$$\begin{aligned} K_2 &= \{a \in A : D(b)a \in L \,\forall\, b \in A\} \\ K_4 &= \{a \in A : D(a)b \in L \,\forall\, b \in A\}. \end{aligned}$$

Then K_2 and K_4 coincide.

Proof. Let a be an element of K_2. By polarisation, $D(b, c)a$ lies in L for all b and c in A. In particular,

$$D(b, a)a = D(a)b$$

lies in L for all b in A. Therefore a lies in K_4. Conversely, suppose that a lies in K_4 and let b be an element of A. Then, using Corollary 4.12,

$$D(b)a^3 = 2\{\{b\,b\,a\}\,a\,a\} - \{a\,\{b\,b\,a\}\,a\}$$

lies in L. Therefore, a^3 lies in K_2. By Proposition 4.9, K_2 is a norm-closed subtriple of A. Therefore, by [18], Lemma 1.2, a lies in K_2. This completes the proof. ∎

The results obtained for the spaces K_2 and K_4 can now be applied to the study of another space, K_3, defined in Proposition 4.14.

Proposition 4.14 *Let A be a JB*-triple, let L be a norm-closed subspace of A and define sets K_2, K_3 and K_4 by*

$$\begin{aligned} K_2 &= \{a \in A : D(b)a \in L \,\forall\, b \in A\}, \\ K_3 &= \{a \in A : Q(b)a \in L \,\forall\, b \in A\}, \\ K_4 &= \{a \in A : D(a)b \in L \,\forall\, b \in A\}. \end{aligned}$$

Then,

$$K_3 = \{a \in A : Q(b,c)a \in L \,\forall\, b, c \in A\},$$

$$K_3 \subseteq K_2 = K_4 \subseteq L$$

and K_3 is a norm-closed ideal of A.

Proof. It is clear that K_3 is a norm-closed subspace of A. By polarisation,

$$K_3 = \{a \in A : Q(b,c)a \in L \,\forall\, b, c \in A\}.$$

In particular,

$$Q(b,a)a = D(a)b$$

lies in L for all b in A. Therefore a lies in K_4, which coincides with K_2 and is a subset of L by Proposition 4.13 and Proposition 4.9. Let a be an element of K_3 and let b, c be elements of A. Then

$$Q(b)\{c\,a\,a\} = 2\{\{a\,c\,b\}\,a\,b\} - \{a\,c\,\{b\,a\,b\}\}$$

lies in L. Therefore $\{c\,a\,a\}$ lies in K_3 for all c in A and a in K_3. It follows by polarisation that K_3 is a norm-closed ideal of A. ∎

It is now possible to prove the main theorem of this section.

Theorem 4.15 *Let A be a JB*-triple, let L be a norm-closed subspace of A, let $k_n(L)$*

be the norm central kernel of L *and let* K_3 *be the subset*

$$K_3 = \{a \in A : Q(b)a \in L \,\forall\, b \in A\}.$$

Then

$$k_n(L) = K_3.$$

Proof. By Proposition 4.14, K_3 is a norm-closed ideal of A contained in L. Hence K_3 is a subset of $k_n(L)$. Let J be a norm-closed ideal of A contained in L. Then for a in J and b in A, $\{b\,a\,b\}$ lies in J and hence L. Therefore a lies in K_3 and J is a subset of K_3. The result follows. ■

4.3 Central kernels and support spaces

The purpose of this section is to establish Corollary 4.19, an expression for the norm central kernel of the kernel of a functional on a JB*-triple. The main result, Theorem 4.16, which leads to Corollary 4.19, is more general than is required here, but is of independent interest [37].

Recall from Theorem 2.17 that for each element x of the pre-dual A_* of the JBW*-triple A, $e(x)$ denotes the support tripotent of x. Let A be a JBW*-triple and let L be a subset of the predual A_*. The *support space* $s(L)$ of L is defined to be the w*-closure of the linear span of the support tripotents of the elements of L [38]. It can be shown that the w*-closed inner ideal $s(L)^\perp$ is given by

$$s(L)^\perp = \bigcap \{A_0(e(x)) : x \in L\}.$$

Theorem 4.16 *Let* A *be a JBW*-triple with predual* A_**, and let* L *be a subset of* A_* *with support space* $s(L)$ *and topological annihilator* L°*. Then the annihilator* $s(L)^\perp$ *and the kernel* $\operatorname{Ker} s(L)$ *of* $s(L)$ *satisfy*

$$s(L)^\perp \subseteq \operatorname{Ker} s(L) \subseteq L^\circ.$$

The central kernels $k(s(L)^{\perp})$, $k(\mathrm{Ker}\, s(L))$ and $k(L^{\circ})$ of the spaces $s(L)$, $\mathrm{Ker}\, s(L)$ and L° respectively satisfy

$$k(s(L)^{\perp}) = k(\mathrm{Ker}\, s(L)) = k(L^{\circ}).$$

Proof. Let a lie in $\mathrm{Ker}\, s(L)$. Then for each element x in L, $P_2(e(x))a$ is zero. Hence,

$$x(a) = x(P_2(e(x))a) = 0,$$

and it follows that $\mathrm{Ker}\, s(L)$ is contained in L°. Now let J be a w*-closed ideal of A contained in L°. Then, for every element a in J and x in L, $\{a\, a\, e(x)\}$ lies in J and $x(\{a\, a\, e(x)\})$ is zero. By [8], Proposition 1.2, a lies in $A_0(e(x))$. Since this is true for all elements x in L it follows that J is contained in $s(L)^{\perp}$. The result is now clear. ■

In particular, if L is the predual of a w*-closed inner ideal J of the JBW*-triple A, it follows from Theorem 4.7 and [35] that J coincides with $s(L)$, that $\mathrm{Ker}\, J$ coincides with $(J_*)^{\circ}$ and that $k(J^{\perp})$ coincides with $k(\mathrm{Ker}\, J)$. We shall be interested in the case when L contains a single element of A_*, and in particular when that element is one of the pure normal functionals x of A, discussed in Theorem 2.19.

Corollary 4.17 *Let A be a JBW*-triple with predual A_*, and let x be a non-zero element of A_* with kernel $\ker x$ and support tripotent $e(x)$. For j equal to $0, 1, 2$, let $A_j(e(x))$, be the Peirce spaces corresponding to $e(x)$. Then,*

$$A_0(e(x)) \subseteq A_0(e(x)) \oplus A_1(e(x)) \subseteq \ker x.$$

For a subspace L of A, let $k(L)$ be the central kernel of L. Then

$$k(A_0(e(x))) = k(A_0(e(x)) \oplus A_1(e(x))) = k(\ker x).$$

Furthermore, $A_0(e(x)) \oplus A_1(e(x))$ and $\ker x$ coincide if and only if x is a scalar multiple of an element of $\partial_e A_{*,1}$, the set of pure normal functionals of A.

Proof. The first part of the result follows by considering the subset L of A_* in the statement of Theorem 4.16 to be equal to the singleton $\{x\}$. Let x be a scalar multiple of an element $\hat{x}$ of $\partial_e A_{*,1}$. Then, $e(x)$ is minimal and, for all elements a in A,

$$P_2(e(x))a = \hat{x}(a)e(x).$$

If a lies in $\ker x$ then $P_2(e(x))a$ is zero. Hence, a is an element of $A_0(e(x)) \oplus A_1(e(x))$. Conversely, if $\ker x$ and $A_0(e(x)) \oplus A_1(e(x))$ agree, then any element a of A may be written

$$a = \hat{x}(a)e(x) - (a - \hat{x}(a)e(x)).$$

Since $a - \hat{x}(a)e(x)$ lies in $\ker x$, it follows that $P_2(e(x))a$ is equal to $\hat{x}(a)e(x)$. Thus $e(x)$ is minimal and $\hat{x}$ lies in $\partial_e A_{*,1}$ as required. ■

Let $\mathcal{I}_{\mathrm{n}}(A)$ be the complete lattice of norm-closed inner ideals of the JB*-triple A. When A is a JBW*-triple, let $\mathcal{I}(A)$ denote the complete lattice of w*-closed inner ideals.

Let A be a JB*-triple and let J be an element of $\mathcal{I}_{\mathrm{n}}(A)$. By separate w*-continuity of the triple product, $J \mapsto \bar{J}^{w*}$ is a map from $\mathcal{I}_{\mathrm{n}}(A)$ to $\mathcal{I}(A^{**})$ for which the map $K \mapsto K \cap A$ from $\mathcal{I}(A^{**})$ onto $\mathcal{I}_{\mathrm{n}}(A)$ is an inverse. It is important to note that this map need not be injective since $\overline{K \cap A}^{w*}$ may be a proper subspace of K.

Lemma 4.18 *Let A be a JB*-triple and let L be a w*-closed subset of the bidual A^{**}. Then the central kernel $k(L)$ of L and the norm central kernel $k_n(L \cap A)$ of the subset $L \cap A$ of A satisfy the equation*

$$k(L) \cap A = k_n(L \cap A).$$

Proof. Since $k(L)$ is contained in L, $k(L) \cap A$ is a norm-closed ideal contained in $L \cap A$ and hence $k(L) \cap A$ is contained in $k_n(L \cap A)$. Conversely, $k_n(L \cap A)$ and hence its w*-closure are ideals contained in L and hence in $k(L)$. ■

Corollary 4.19 *Let A be a JB*-triple with dual A^* and bidual A^{**}. For any x in A^*, let $\ker x$ be the kernel of x in A, let $k_n(\ker x)$ be the norm central kernel of x in A, let $e(x)$ be the support tripotent of x in A^{**}, let $A_0^{**}(e(x))$ be the Peirce-0 space of $e(x)$ in A^{**}, let $k(A_0^{**}(e(x)))$ be the central kernel of $A_0^{**}(e(x))$ in A^{**} and let $k_n(A_0^{**}(e(x))\cap A)$ be the norm central kernel of $A_0^{**}(e(x)) \cap A$ in A. Then*

$$k_n(\ker x) = k(A_0^{**}(e(x))) \cap A = k_n(A_0^{**}(e(x)) \cap A).$$

Proof. The result follows from Corollary 4.17 and Lemma 4.18. ∎

Corollary 4.19 gives a third characterisation of the primitive spectrum $\operatorname{Prim} A$ of a JB*-triple A, in terms of the set of pure functionals $\partial_e A_1^*$ of A, namely, that

$$\operatorname{Prim} A = \{k(A_0^{**}(e(x))) \cap A : x \in \partial_e A_1^*\}.$$

4.4 The spectrum

This section obtains some additional results about the spectrum of a JB*-triple A. Recall that the spectrum of an arbitrary Banach space A was defined in Section 3.3 using the set $\partial_e A_1^*$ of pure functionals of A. The main result of this section, Theorem 4.24 is a topological result connecting $\partial_e A_1^*$ to the spectrum in the JB*-triple case.

Lemma 4.20 *Let A be a JB*-triple and let L be an L-summand in A^*. Then $\overline{L}^{w*}$ is an L-summand of A^*.*

Proof. Since L° is a norm-closed ideal in A^{**}, $A \cap L^\circ$ is a norm-closed ideal in A. Since $A \cap L^\circ$ coincides with $L_\circ$, the result follows. ∎

Corollary 4.21 *Let A be a JB*-triple, let x be an element of $\partial_e A_1^*$, the set of pure functionals of A, let L_x be the smallest L-summand of A^* containing x and let K_x be the smallest w*-closed L-summand of A^* containing x. Then K_x coincides with the w*-closure of L_x.*

Proposition 4.22 *Let A be a JB*-triple, let $\partial_e A_1^*$ be the set of pure functionals of A and let S be a subset of the spectrum,* $\operatorname{Spec} A$ *of A. For a pure functional x of A, let L_x denote the smallest L-summand containing x, let J_x denote the norm central kernel $k_n(\ker x)$ of x in A, let J^x denote the central kernel $k(\ker x)$ of x in A^{**} and let $\Theta : \partial_e A_1^* \mapsto \operatorname{Spec} A$ denote the map $x \mapsto J^x$. Then:*

(i) $(\bigcap\{J_x : J^x \in S\})^\circ = \overline{\operatorname{lin}\{L_x : J^x \in S\}}^{w*}$;

(ii) $\partial_e(\bigcap\{J_x : J^x \in S\})_1^\circ \subseteq \overline{\bigcup\{\partial_e(L_x)_1 : J^x \in S\}}^{w*} = \overline{\Theta^{-1}(S)}^{w*}$.

Proof. For x in $\partial_e A_1^*$, let K_x denote the smallest w*-closed L-summand of A^* containing x. By [21], Lemma 3.3 and [21], Lemma 3.4 (a),

$$\begin{aligned}\partial_e\left(\cap\{J_x : J^x \in S\}\right)_1^\circ &\subseteq \overline{\left(\cup\{J_x^\circ : J^x \in S\}\right) \cap \partial_e A_1^*}^{w*}\\ &= \overline{\cup\{\partial_e(K_x)_1^\circ : J^x \in S\}}^{w*}\\ &\subseteq \overline{\cup\{\partial_e(L_x)_1 : J^x \in S\}}^{w*}\\ &= \overline{\Theta^{-1}(S)}^{w*}.\end{aligned}$$

Clearly $\overline{\operatorname{lin}\{L_x : J^x \in S\}}^{w*}$ is a subset of $(\bigcap\{J_x : J^x \in S\})^\circ$. Conversely, by the Krein-Milman Theorem

$$(\cap\{J_x : J^x \in S\})_1^\circ \subseteq \overline{\operatorname{conv}\left(\cup\{\partial_e(L_x)_1 : J^x \in S\}\right)}^{w*} \subseteq \overline{\operatorname{lin}\{L_x : J^x \in S\}}^{w*}$$

and the result follows. ∎

Corollary 4.23 *Let A be a JB*-triple, let $\partial_e A_1^*$ be the set of pure functionals of A, let S be a subset of the spectrum* $\operatorname{Spec} A$ *and let $\overline{S}^{\mathfrak{J}'}$ be the closure of S in the structure topology $\mathfrak{J}'$. For an element x of $\partial_e A_1^*$, let L_x denote the smallest L-summand containing x, let J^x denote the central kernel $k(\ker^* x)$ of x in A^{**} and let $\Theta : \partial_e A_1^* \mapsto$* $\operatorname{Spec} A$ *denote the map $x \mapsto J^x$. Then, for an element x of $\partial_e A_1^*$, the following are equivalent:*

(i) $J^x \in \overline{S}^{\mathfrak{J}'}$;

(ii) $\overline{\Theta^{-1}(S)}^{w*} \cap L_x \neq \emptyset$;

(iii) $\partial_e(L_x)_1 \subseteq \overline{\Theta^{-1}(S)}^{w*}$.

Proof. (i)⇒(iii): Suppose that J^x is an element of $\overline{S}^{\mathfrak{J}'}$. Then K_x, the smallest w*-closed L-summand of A^* containing x, is contained in $(\cap_{J^x \in S} J_x)^\circ$ and by Proposition 4.22 (ii), x is contained in $\overline{\Theta^{-1}(S)}^{w*}$. Let y be an element of $\partial_e(L_x)_1$. Then J^y and J^x coincide and y lies in $\overline{\Theta^{-1}(S)}^{w*}$.

(iii)⇒(ii): This is true by definition.

(ii)⇒(i): Let y lie in $\overline{\Theta^{-1}(S)}^{w*} \cap L_x$. Then $\cap_{J^x \in S} J_x$ is annihilated by y and is therefore contained in J_y. By Lemma 3.25, $\overline{S}^{\mathfrak{J}'}$ contains J^y which equals J^x. ∎

Theorem 4.24 *Let A be a JB*-triple, let $\partial_e A_1^*$ be the set of pure functionals of A, equipped with the relative w*-topology and let* Spec A *be the spectrum of A equipped with the structure topology $\mathfrak{J}'$. For each element x in $\partial_e A_1^*$, let J_x be the norm central kernel $k_n(\ker x)$ of x. Then the map $\Theta : \partial_e A_1^* \mapsto \mathrm{Spec}\, A$ given by $x \mapsto J^x$ is open.*

Proof. Let U be a w*-open subset of $\partial_e A_1^*$. Assume that there exists an element J^x in $\overline{\mathrm{Spec}\, A \backslash \Theta(U)}^{\mathfrak{J}'}$ such that x lies in U. Then x lies in $\overline{\Theta^{-1}(\mathrm{Spec}\, A \backslash \Theta(U))}^{w*}$. This is a closed subset of $\partial_e A_1^* \backslash U$. By contradiction, $\mathrm{Spec}\, A \backslash \Theta(U)$ is closed. ∎

The result of [21], Theorem 3.5 can be immediately recovered from Theorem 4.24. In the special case of a C*-algebra, these results are similar to [25], 3.4.2, 3.4.10 and 3.4.11, but with the larger set of pure functionals replacing the pure states.

4.5 The primitive spectrum

In this section it is shown that for each element a in the JB*-triple A, the map $P \mapsto \|a + P\|$ is lower semi-continuous on the primitive spectrum $\mathrm{Prim}\, A$, of A. This has the important consequence that $\mathrm{Prim}\, A$ is a locally compact space. The main results of this section can be found in [17], [20].

Lemma 4.25 *Let A be a Banach space, let $\{I_\lambda : \lambda \in \Lambda\}$ be a non-empty collection of closed subspaces of A such that for any λ_1, λ_2 in Λ, there exists λ_3 in Λ such that I_{λ_1}, I_{λ_2} are subsets of I_{λ_3}. Then $I = \overline{\cup_{\lambda\in\Lambda} I_\lambda}$ is a closed subspace of A, and, for each element a in A,*

$$\|a + I\| = \inf_{\lambda\in\Lambda} \|a + I_\lambda\|.$$

Proof. Clearly $\cup_{\lambda\in\Lambda} I_\lambda$ is a subspace of A. Let $\varepsilon > 0$. Then, there exists an element b in I such that

$$\|a + b\| < \|a + I\| + \varepsilon/2.$$

For some λ in Λ, I_λ contains an element c such that $\|b - c\| < \varepsilon/2$. Thus

$$\|a + I_\lambda\| \leq \|a + c\| \leq \|a + b\| + \|c - b\| < \|a + I\| + \varepsilon.$$

Since, for all λ in Λ, we also have,

$$\|a + I\| \leq \|a + I_\lambda\|$$

the result follows. ■

Lemma 4.26 *Let A be a JB*-triple, let $\{J_\lambda : \lambda \in \Lambda\}$ be a family of norm-closed ideals of A and let J be the norm-closed ideal $\cap_{\lambda\in\Lambda} J_\lambda$. Then*

$$\|a + J\| = \sup_{\lambda\in\Lambda} \|a + J_\lambda\|.$$

Proof. The natural triple homomorphism of A/J into $\bigoplus_{\lambda\in\Lambda}^{\infty} A/J_\lambda$ is injective and therefore an isometry by [10], Lemma 1. ■

Lemma 4.27 will be required in the proof of Theorem 5.14.

Lemma 4.27 *Let A be a Banach space, let a be an element of A and let I be a proper M-ideal of A with topological annihilator I°. Then there exists an element x in $\partial_e I_1^\circ$,* *the set of extreme points of the unit ball I_1° of I°, with norm central kernel J_x, such that*

$$x(a) = \|a + I\| = \|a + J_x\|.$$

Proof. First consider the case in which I is zero. Without loss of generality, assume that a has unit norm. Then the w*-closed face

$$\{a\}' = \{x \in A_1^* : x(a) = 1\}$$

of A_1^* is non-empty by the Hahn-Banach theorem and possesses an extreme point by the Krein-Milman theorem. Thus, there exists an element x in $\partial_e A_1^*$, the set of pure functionals of A, such that $x(a)$ equals 1. Now, for all elements b in J_x,

$$\|a\| = x(a) = x(a + b) \leq \|a + b\|.$$

Thus

$$\|a\| \leq \|a + J_x\| \leq \|a\|.$$

For the general case, the above shows that there exists an element y in $\partial_e(A/I)_1^*$ with norm central kernel J_y in A/I such that

$$y(a + I) = \|a + I\| = \|(a + I) + J_y\|.$$

By Lemma 3.8, there an element x in $\partial_e I_1^\circ$ with norm central kernel J_x in A such that

$$x(a) = y(a + I).$$

By Lemma 3.6, for all b in A, $b + J_x$ lies in J_y. Thus,

$$\|a + I\| = \|(a + I) + J_y\| \leq \inf\{\|(a + b) + I\| : b \in J_x\} \leq \|a + J_x\| \leq \|a + I\|.$$

The result follows. ■

A *filter* on a set S is a non-empty collection $\mathcal{F}$ of non-empty subsets of X such that

$$F_1, F_2 \in \mathcal{F} \Rightarrow F_1 \cap F_2 \in \mathcal{F}$$
$$F \in \mathcal{F},\ F \subseteq F' \Rightarrow F' \in \mathcal{F}.$$

If X is a topological space, a point x of X is said to be a *cluster point* of the filter $\mathcal{F}$ on X if each F in $\mathcal{F}$ has non-empty intersection with each neighbourhood of x. The point x is a cluster point of $\mathcal{F}$ if and only if x lies in $\cap\{\overline{F} : F \in \mathcal{F}\}$. A topological space is compact if and only if each filter has a cluster point [77], Theorem 17.4.

Proposition 4.28 *Let A be a JB*-triple with primitive spectrum $\operatorname{Prim} A$, equipped with the Jacobson topology $\mathfrak{J}$. For an element a in A, and $\alpha > 0$, let X be the set*

$$X = \{P \in \operatorname{Prim} A : \|a + P\| \geq \alpha\}.$$

Then X is $\mathfrak{J}$-compact.

Proof. Let $\mathcal{F}$ be a filter on X. For F in $\mathcal{F}$, define J_F to be the norm-closed ideal $\cap_{P\in F} P$. Then, for any element P in F,

$$\|a + J_F\| \geq \|a + P\| \geq \alpha$$

For elements F_1, F_2 in $\mathcal{F}$, $F_1 \cap F_2$ lies in $\mathcal{F}$. Thus, J_{F_1}, J_{F_2} are contained in $J_{F_1 \cap F_2}$. By Lemma 4.25 and Theorem 3.4,

$$J = \overline{\bigcup\{J_F : F \in \mathcal{F}\}}^n = \bigvee\{J_F : F \in \mathcal{F}\}$$

is a closed ideal of A,

$$\|a + J\| = \inf_{F\in\mathcal{F}} \|a + J_F\| \geq \alpha.$$

and J is a proper ideal of A. By Lemma 4.27, there exists an x in $\partial_e J_1^\circ$ with norm central kernel J_x such that

$$\alpha \leq \|a + J\| \leq \|a + J_x\|.$$

Therefore J_x lies in X. Let $\operatorname{hull} J_F$ be the hull of J_F. Since x annihilates J, for each element F in $\mathcal{F}$,

$$J_x \in \operatorname{hull} J_F = \overline{F}^{\mathfrak{J}}.$$

Therefore, J_x is an element of $\cap\{\overline{F}^{\mathfrak{J}} : F \in \mathcal{F}\}$ and J_x is a cluster point of $\mathcal{F}$. The result follows. ■

Proposition 4.29 *Let A be a JB*-triple and let $\mathcal{ZI}_{\rm n}(A)$ be the complete lattice of norm-closed ideals of A. For each element a in A define the mapping $\rho_a : \mathcal{ZI}_{\rm n}(A) \to [0, \infty)$ by*

$$\rho_a(I) = \|a + I\|.$$

Then, for all elements a in A, with respect to the Jacobson topology, the function ρ_a is lower semi-continuous on the primitive spectrum $\operatorname{Prim} A$.

Proof. Let α be a real number and put

$$\begin{aligned} S &= \rho_a|_{\operatorname{Prim} A}^{-1}((-\infty, \alpha]) \\ &= \{P \in \operatorname{Prim} A : \|a + P\| \leq \alpha\}. \end{aligned}$$

Let Q lie in the closure of S in $\operatorname{Prim} A$. Then $\bigcap S$ is contained in Q, and, hence,

$$\begin{aligned} \|a + Q\| &\leq \|a + \cap S\| \\ &= \sup\{\|a + P\| : P \in S\} \\ &\leq \alpha. \end{aligned}$$

Thus, Q is contained in S, and S is closed. Therefore $\rho_a^{-1}(\alpha, \infty)$ is open and ρ_a is lower semi-continuous, as required. ■

Corollary 4.30 *Let A be a JB*-triple with primitive spectrum* $\mathrm{Prim}\, A$*. Let β be the collection of subsets of* $\mathrm{Prim}\, A$ *of the form*

$$\{P \in \mathrm{Prim}\, A : \|a + P\| > 1\}$$

for a in A. Then β is a base for the Jacobson topology on $\mathrm{Prim}\, A$.

Proof. By Proposition 4.29, every element of β is an open subset of $\mathrm{Prim}\, A$. Let U be an open subset of $\mathrm{Prim}\, A$ and let I be the norm-closed ideal such that U has complement $\mathrm{hull}\, I$. Then, for any element P_0 of U, I is not a subset of P_0 and there exists an element a in I such that

$$\|a + P_0\| > 1.$$

For Q in $\mathrm{hull}\, I$, $\|a + Q\|$ takes the value zero. Thus

$$P_0 \in \{Q \in \mathrm{Prim}\, A : \|a + Q\| > 1\} \subseteq U$$

and β is a base for the Jacobson topology on $\mathrm{Prim}\, A$. ■

Proposition 4.31 *Let A be a JB*-triple with primitive spectrum* $\mathrm{Prim}\, A$*. Then* $\mathrm{Prim}\, A$ *is locally compact in the Jacobson topology.*

Proof. Let P be a primitive ideal and let U be an open neighbourhood of P in $\mathrm{Prim}\, A$. Then $\mathrm{Prim}\, A \backslash U$ coincides with the hull $\mathrm{hull}\, J$ of some norm-closed ideal J of A such that J is not a subset of P. Choose an element a in $J \backslash P$ and let ρ_a be as defined in 4.29. Then $\|a + P\|$ is non-zero and ρ_a is zero on $\mathrm{hull}\, J$. Define

$$V = \{Q \in \mathrm{Prim}\, A : \|a + Q\| > 1/2\|a + P\|\}$$
$$C = \{Q \in \mathrm{Prim}\, A : \|a + Q\| \geq 1/2\|a + P\|\}.$$

Then

$$P \in V \subseteq C \subseteq U$$

and U contains the compact neighbourhood C of P as required. ■

4.6 Topologies on the complete lattice of ideals

In this section it is shown that the lower, Scott and Lawson topologies (Section 3.2) on the complete lattice of M-ideals of a JB*-triple can be classified in a way which will be useful in the construction of continuous cross-sections over spaces of ideals in Chapters 5 and 6.

The first result, Proposition 4.32, reduces to [53], Theorem 7.2 in the case of a C*-algebra. See also [42], Theorem 2.2.

Proposition 4.32 *Let A be a JB*-triple and let $\mathcal{ZI}_{\mathrm{n}}(A)$ be the complete lattice of norm-closed ideals of A. For each element a in A define the mapping $\rho_a : \mathcal{ZI}_{\mathrm{n}}(A) \mapsto [0, \infty)$ by*

$$\rho_a(I) = \|a + I\|.$$

Then the following results hold.

(i) *The lower topology is the weakest topology for which, for all elements a in A, ρ_a is lower semi-continuous on $\mathcal{ZI}_{\mathrm{n}}(A)$.*

(ii) *The Scott topology is the weakest topology for which, for all elements a in A, ρ_a is upper semi-continuous on $\mathcal{ZI}_{\mathrm{n}}(A)$.*

(iii) *The Lawson topology is the weakest topology for which, for all elements a in A, ρ_a is continuous on $\mathcal{ZI}_{\mathrm{n}}(A)$.*

Proof. Let $\mathfrak{W}$ and $\mathfrak{S}$ denote the lower and Scott topologies on $\mathcal{ZI}_{\mathrm{n}}(A)$ respectively and let $\mathfrak{J}$ denote the Jacobson topology on $\mathrm{Prim}\, A$, the primitive spectrum of A.

(i) Let a be an element of A, let J be an element of $\mathcal{ZI}_{\mathrm{n}}(A)$ and let $\varepsilon > 0$. Since

$$\|a + J\| = \sup_{P \in \mathrm{hull}\, J} \|a + P\|,$$

there exists an element P in $\operatorname{hull} J$ such that

$$\|a + J\| - \varepsilon/2 < \|a + P\|.$$

By Proposition 4.29, ρ_a is $\mathfrak{J}$-lower semi-continuous on $\operatorname{Prim} A$. Thus the set V defined by

$$\begin{aligned} V &= \{Q \in \operatorname{Prim} A : \|Q + a\| > \|a + J\| - \varepsilon\} \\ &= \rho_a^{-1}(\|a + J\| - \varepsilon, \infty) \end{aligned}$$

is $\mathfrak{J}$-open in $\operatorname{Prim} A$ and contains P. Define a subset $U(\emptyset, \{V\})$ of $\mathcal{ZI}_{\rm n}(A)$ by

$$U(\emptyset, \{V\}) = \{K \in \mathcal{ZI}_{\rm n}(A) : V \cap \operatorname{hull} K \neq \emptyset\}.$$

Then $U(\emptyset, \{V\})$ is a $\mathfrak{W}$-open neighbourhood of J and, for an element K in $U(\emptyset, \{V\})$, there exists an element Q in $V \cap \operatorname{hull} K$ such that

$$\|a + K\| \geq \|a + Q\| > \|a + J\| - \varepsilon.$$

Thus ρ_a is $\mathfrak{W}$ lower semi-continuous at J.

Now, let $\mathfrak{T}$ be the weakest topology such that for all elements a in A, ρ_a is lower semi-continuous. We have shown that $\mathfrak{T}$ is weaker than $\mathfrak{W}$. Let V be a $\mathfrak{W}$-open subset of $\mathcal{ZI}_{\rm n}(A)$ and let I be an element of V. Then V contains a $\mathfrak{W}$-open neighbourhood U of the form

$$U = U(\{J_1, \ldots, J_n\}) = \bigcap_{j=1}^{n} \mathcal{ZI}_{\rm n}(A) \backslash \uparrow J_j$$

for some $J_1, \ldots, J_n$ in $\mathcal{ZI}_{\rm n}(A)$. Thus, for j equal to $1, \ldots, n$, we may choose an element a_j in $J_j \backslash I$. Then,

$$I \in \cap_{j=1}^{n} \rho_{a_j}^{-1}(0, \infty) = \bigcap_{j=1}^{n} \{J \in \mathcal{ZI}_{\rm n}(A) : a_j \notin J\} \subseteq U \subseteq V.$$

Thus, every element of V has a $\mathfrak{T}$-open neighbourhood in V. It follows that $\mathfrak{W}$ and $\mathfrak{T}$ are the same topology.

(ii) Let a be an element of A, let J be an element of $\mathcal{ZI}_{\rm n}(A)$ and let $\varepsilon > 0$. Define C to be the subset of $\operatorname{Prim} A$ given by

$$C = \{P \in \operatorname{Prim} A : \|a + P\| \geq \|a + J\| + \varepsilon\}.$$

Then, by Proposition 4.28, C is $\mathfrak{J}$-compact. Let $U(C, \emptyset)$ be the $\mathfrak{S}$-open set defined by

$$\begin{aligned} U(C, \emptyset) &= \{K \in \mathcal{ZI}_{\rm n}(A) : C \cap \operatorname{hull} K = \emptyset\} \\ &= \{K \in \mathcal{ZI}_{\rm n}(A) : \|a + P\| < \|a + J\| + \varepsilon \quad \forall P \in \operatorname{hull} K\}. \end{aligned}$$

Then, $U(C, \emptyset)$ is a $\mathfrak{S}$-open neighbourhood of J, and, for K in $U(C, \emptyset)$,

$$\|a + K\| = \sup_{P \in \operatorname{hull} K} \|a + P\| \leq \|a + J\| + \varepsilon.$$

Therefore ρ_a is $\mathfrak{S}$ upper semi-continuous at J.

Now let $\mathfrak{T}$ be the weakest topology such that, for all elements a in A, ρ_a is upper semi-continuous. We have shown that $\mathfrak{T}$ is weaker than $\mathfrak{S}$. Let V be a $\mathfrak{S}$-open subset of $\mathcal{ZI}_{\rm n}(A)$ and let Q be its complement. By [53], Theorem 3.7, Q is $\mathfrak{W}$-compact and lower, that is to say, if $I \leq J$ for some I in $\mathcal{ZI}_{\rm n}(A)$ and J in Q then I lies in Q. Let I be an element of V. Using (i), for an element a in I, we may define the $\mathfrak{W}$-open set $U(a)$ by

$$U(a) = \{J \in \mathcal{ZI}_{\rm n}(A) : \|a + J\| > 1\} = \rho_a^{-1}(1, \infty).$$

Since Q is lower, for J in Q we may choose an element a in $I \backslash J$ such that $\|a + J\| > 1$. Thus the set of $\{U(a) : a \in A\}$ is an $\mathfrak{W}$-open cover for Q. Let $a_1, \ldots, a_n$ be elements of A be such that $U(a_1), \ldots, U(a_n)$ is a finite subcover of Q. Then,

$$I \in \bigcap_{j=1}^{n} \rho_{a_j}^{-1}(-\infty, 1) \subseteq V.$$

Thus, every element of V has a $\mathfrak{T}$-open neighbourhood in V. It follows that $\mathfrak{S}$ and $\mathfrak{T}$ are the same topology.

(iii) Immediate from (i) and (ii). ■

Lemma 4.33 presents a further characterisation of the Lawson topology which will be useful in the sequel.

Lemma 4.33 *Let A be a JB*-triple and let $\mathcal{ZI}_{\mathrm{n}}(A)$ be the complete lattice of norm-closed ideals of A equipped with the Lawson topology. Then, the net (J_λ) in $\mathcal{ZI}_{\mathrm{n}}(A)$ converges to an element J in $\mathcal{ZI}_{\mathrm{n}}(A)$ if and only if, for all elements a in A, the net $(\|a + J_\lambda\|)$ converges to $\|a + J\|$.*

Proof. For J in $\mathcal{ZI}_{\mathrm{n}}(A)$, define a mapping $N_J : a \mapsto \|a + J\|$. It follows from Proposition 4.32 (iii) that the map $N : J \mapsto N_J|_{A_1}$ is a homeomorphism from $\mathcal{ZI}_{\mathrm{n}}(A)$ onto $N(A)$, the image of N in the product space $[0, 1]^{A_1}$, equipped with the topology of pointwise convergence. This completes the proof. ■

Lemma 4.34 *Let A be a JB*-triple with primitive spectrum* Prim A*. Then the Jacobson topology $\mathfrak{J}$ is Hausdorff if and only if it coincides with the restriction of the Lawson topology $\mathfrak{L}$ to* Prim A*.*

Proof. Suppose that Prim A is $\mathfrak{J}$-Hausdorff and let C be a $\mathfrak{L}$-closed subset of $\mathcal{ZI}_{\mathrm{n}}(A)$. Let (P_λ) be a net in $C \cap \mathrm{Prim}\, A$, $\mathfrak{J}$-converging to some element P in Prim A. Since Prim A is $\mathfrak{J}$-Hausdorff, $\rho_a|_{\mathrm{Prim}\, A}$ is Jacobson continuous and, for all a in A, the net $(\rho_a(P_\lambda))$ converges to $\rho_a(P)$. Thus (P_λ) $\mathfrak{L}$-converges to P and P lies in $C \cap \mathrm{Prim}\, A$. It follows that if U is $\mathfrak{L}$-open in $\mathcal{ZI}_{\mathrm{n}}(A)$ then $U \cap \mathrm{Prim}\, A$ is $\mathfrak{J}$-open in Prim A. The converse is immediate since $\mathfrak{L}$ is Hausdorff. ■

4.7 Primitive spectrum and centralizer

This section returns to the complete regularisation of the primitive spectrum and the Dauns-Hofmann theorem (Theorem 3.26) to find an explicit construction of the

Stone-Čech compactification of the primitive spectrum. The results of this section will be used in Chapter 5 to interpret the action of the centralizer on a quasi-standard JB*-triple (Corollary 5.18). Results of a similar nature to the results in this section have been obtained for algebras in [22], III.6.

Lemma 4.35 *Let A be a JB*-triple with centralizer $Z(A)$, let $\mathcal{ZI}_{\mathrm{n}}(A)$ be the complete lattice of norm-closed ideals of A and let $\mathcal{ZI}_{\mathrm{n}}(Z(A))$ be the complete lattice of norm-closed ideals of $Z(A)$. For each element J of $\mathcal{ZI}_{\mathrm{n}}(A)$, let $J^{\triangledown}$ be the subset of $Z(A)$ defined by*

$$J^{\triangledown} = \{T \in Z(A) : TA \subseteq J\}.$$

and for each element I of $\mathcal{ZI}_{\mathrm{n}}(Z(A))$, let $I_{\triangle}$ be the subset of A defined by

$$I_{\triangle} = IA = \{Ta : T \in I, a \in A\}.$$

Then the following results hold.

(i) *The mapping $J \mapsto J^{\triangledown}$ is an order preserving mapping from $\mathcal{ZI}_{\mathrm{n}}(A)$ into $\mathcal{ZI}_{\mathrm{n}}(Z(A))$.*

(ii) *The mapping $I \mapsto I_{\triangle}$ is an order preserving mapping from $\mathcal{ZI}_{\mathrm{n}}(Z(A))$ into $\mathcal{ZI}_{\mathrm{n}}(A)$.*

(iii) *For each element J of $\mathcal{ZI}_{\mathrm{n}}(A)$ and each element I of $\mathcal{ZI}_{\mathrm{n}}(Z(A))$,*

$$J \supseteq (J^{\triangledown})_{\triangle}, \qquad I \subseteq (I_{\triangle})^{\triangledown}.$$

(iv) *For each family $\{J_\lambda : \lambda \in \Lambda\}$ of elements of $\mathcal{ZI}_{\mathrm{n}}(A)$,*

$$\cap\{J_\lambda^{\triangledown} : \lambda \in \Lambda\} = (\cap\{J_\lambda : \lambda \in \Lambda\})^{\triangledown}.$$

Proof. Let I be an element of $\mathcal{ZI}_{\mathrm{n}}(Z(A))$. By the Cohn factorisation theorem ([26], B.7.1), $I_{\triangle}$ is a closed subspace of A, and it follows that $I_{\triangle}$ is an ideal of A. The rest of the result follows easily from the definitions. ■

Lemma 4.36 *Let A be a JB*-triple with centralizer $Z(A)$. Let x be an element of $\partial_e A_1^*$, the set of pure functionals of A, let $\check{x}$ be the character of $Z(A)$ defined for T in $Z(A)$ by*

$$T^* x = \check{x}(T)x,$$

let $k_n(\ker x)$ be the norm central kernel of x, and let $\ker \check{x}$ be the kernel of $\check{x}$. Then, adopting the notation of Lemma 4.35, $P \mapsto P^\triangledown$ is a mapping from $\operatorname{Prim} A$, the primitive spectrum of A into $\operatorname{Prim} Z(A)$, the primitive spectrum of $Z(A)$ such that

$$k_n(\ker x)^\triangledown = \ker \check{x}.$$

Proof. Let x be an element of $\partial_e A_1^*$, let T be an element of $k_n(\ker x)^\triangledown$ and choose an element a in A such that $x(a)$ is non-zero. Then,

$$\check{x}(T)x(a) = x(Ta) = 0,$$

which implies that T lies in $\ker \check{x}$. Conversely, let T be an element of $\ker \check{x}$. Then the ideal TA is contained in $\ker x$ and hence in $k_n(\ker x)$. As a consequence, for every element P in $\operatorname{Prim} A$, $P^\triangledown$ lies in $\operatorname{Prim} Z(A)$. ■

Lemma 4.37 *Let A be a JB*-triple, and adopt the notation of Lemma 4.35. For an element J of $\mathcal{ZI}_{\mathrm{n}}(A)$, let $\operatorname{hull}_A J$ be the hull of J in $\operatorname{Prim} A$ and for an element I of $\mathcal{ZI}_{\mathrm{n}}(Z(A))$, let $\operatorname{hull}_{Z(A)} I$ denote the hull of I in $\operatorname{Prim} Z(A)$. Let $\mathfrak{J}_A$ be the Jacobson topology on $\operatorname{Prim} A$. Then:*

(i) *for each element J of $\mathcal{ZI}_{\mathrm{n}}(A)$,*

$$\operatorname{hull}_{Z(A)} J^\triangledown = \overline{\{P^\triangledown : P \in \operatorname{hull}_A J\}}^{\mathfrak{J}_A};$$

(ii) *for each element I of $\mathcal{ZI}_{\mathrm{n}}(Z(A))$,*

$$\operatorname{hull}_A I_\triangle = \{P \in \operatorname{Prim} A : P^\triangledown \in \operatorname{hull}_{Z(A)} I\}.$$

Proof. Let J be a norm-closed ideal of A. Then,

$$J^{\triangledown} = (\cap \operatorname{hull}_A J)^{\triangledown} = \cap\{P^{\triangledown} : P \in \operatorname{hull}_A J\}.$$

Taking the hull in $\operatorname{Prim} Z(A)$ of both sides, (i) follows.

Let I be a norm-closed ideal of $Z(A)$ and let P be an element of $\operatorname{hull}_A I_{\triangle}$. Then,

$$I \subseteq (I_{\triangle})^{\triangledown} \subseteq P^{\triangledown},$$

and it follows that $P^{\triangledown}$ lies in $\operatorname{hull}_{Z(A)} I$. Conversely, if P is a primitive ideal of A such that $P^{\triangledown}$ lies in $\operatorname{hull}_{Z(A)} I$ then,

$$I_{\triangle} \subseteq (P^{\triangledown})_{\triangle} \subseteq P.$$

Thus, P lies in $\operatorname{hull}_A I_{\triangle}$, thereby proving (ii). ■

Let Y and Z be topological spaces. A map $\phi : Y \mapsto Z$ is said to be an embedding of Y into Z if ϕ is a homeomorphism of Y onto the subset $\phi(Y)$ of Z. The pair (ϕ, Z) is said to be a compactification of Y if Z is a compact Hausdorff space and $\phi(Y)$ is dense in Z. The compactifications (ϕ, Z) and (ϕ', Z') are said to be topologically equivalent if there exists a homeomorphism $\theta : Z \mapsto Z'$ such that ϕ' equals $\phi \circ \theta$.

Theorem 4.38 *Let Y be a completely regular space. Then there exists a compactification $(\beta, \beta Y)$ of Y such that every continuous bounded real valued function on $\beta(Y)$ has a (necessarily unique) extension to βY. This compactification is unique up to topological equivalence.*

The compactification $(\beta, \beta Y)$ in Theorem 4.38 is said to be the Stone-Čech compactification of Y. For more information about the Stone-Čech compactification, see [76],

Proposition 4.39 *Let A be a JB*-triple with centralizer $Z(A)$. Let* $\operatorname{Prim} A$*, and* $\operatorname{Prim} Z(A)$ *be the primitive spectrums of A and $Z(A)$ respectively, equipped with their*

Jacobson topologies. Let $(\gamma, \gamma \operatorname{Prim} A)$ *be the complete regularisation of* $\operatorname{Prim} A$*. In the notation of Lemma 4.35, let* $\triangledown : \operatorname{Prim} A \mapsto \operatorname{Prim} Z(A)$ *be the mapping* $P \mapsto P^{\triangledown}$*. Then there exists a compactification* $(\triangledown', \operatorname{Prim} Z(A))$ *of* $\gamma \operatorname{Prim} A$ *such that*

$$\triangledown' \circ \gamma = \triangledown$$

and $(\triangledown', \operatorname{Prim} Z(A))$ *is topologically equivalent to the Stone-Čech compactification of* $\gamma \operatorname{Prim} A$.

Proof. It follows from Lemma 4.37 that $\triangledown$ is a continuous map from $\operatorname{Prim} A$ into $\operatorname{Prim} Z(A)$. The latter space is compact and Hausdorff, and hence completely regular. By Proposition 3.14, $\triangledown$ induces a continuous mapping $\triangledown' : \gamma \operatorname{Prim} A \mapsto \operatorname{Prim} Z(A)$ such that the mappings $\triangledown' \circ \gamma$ and $\triangledown$ coincide on $\operatorname{Prim} A$. Let $(\beta, \beta \operatorname{Prim} A)$ denote the Stone-Čech compactification of $\gamma \operatorname{Prim} A$. Then, by [77], 19.5, $\triangledown'$ induces a continuous mapping $\triangledown'' : \beta \operatorname{Prim} A \mapsto \operatorname{Prim} Z(A)$ such that

$$\triangledown'' \circ \beta \circ \gamma = \triangledown' \circ \gamma = \triangledown.$$

Using Lemma 4.37,

$$\operatorname{Prim} Z(A) = \operatorname{hull}(\{0\}^{\triangledown}) = \overline{\{P\triangledown : P \in \operatorname{Prim} A\}}^{\mathfrak{J}}.$$

It follows that $\triangledown''(\beta \operatorname{Prim} A)$ is a compact subset of $\operatorname{Prim} Z(A)$ containing a dense subset $\triangledown(\operatorname{Prim} A)$. Hence $\triangledown''$ is surjective.

The continuous surjection $\triangledown''$ induces an injective *-homomorphism $(\triangledown'')^*$ from $Z(A)$ into $C(\beta(\operatorname{Prim} A))$, the space of continuous complex-valued functions on $\beta(\operatorname{Prim} A)$. Let $\partial_e A_1^*$ be the set of pure functionals of A and let $\Psi : \partial_e A_1^* \mapsto \operatorname{Prim} A$ be the mapping taking each element x in $\partial_e A_1^*$ to its norm central kernel $k_n(\ker x)$. For each element f in $C(\beta(\operatorname{Prim} A))$, the function $f \circ \beta \circ \gamma \circ \Psi$ lies in $C^b(\partial_e A_1^*)$, the space of structure-continuous complex-valued bounded functions on $\partial_e A_1^*$. By Theorem 3.26, there exists an element T in $Z(A)$ such that, for all functionals x in $\partial_e A_1^*$,

$$f \circ \iota \circ \gamma \circ \Psi(x)(1 + \ker \check{x}) = T + \ker \check{x}.$$

Then,

$$((\bigtriangledown'')^* T)(\iota \circ \gamma(k_n(\ker x)))(1 + \ker \check{x}) = T + k_n(\ker x)^{\bigtriangledown},$$

and, on a dense subset of $\beta \operatorname{Prim} A$,

$$((\bigtriangledown'')^* T) = f.$$

Therefore $(\bigtriangledown'')^*$ is a *-isomorphism and $\bigtriangledown''$ is a homeomorphism. Since $\bigtriangledown'' \circ \beta$ equals $\bigtriangledown'$ on $\gamma \operatorname{Prim} A$, this implies that $(\bigtriangledown', \operatorname{Prim} Z(A))$ is a compactification of $\gamma \operatorname{Prim} A$, topologically equivalent to $(\beta, \beta \operatorname{Prim} A)$. ■

It will frequently be useful to identify the complete regularisation of the primitive spectrum of a JB*-triple with a set of ideals, as described in Lemma 4.40.

Lemma 4.40 *Let A be a JB*-triple with primitive spectrum* $\operatorname{Prim} A$*, let* $(\gamma, \gamma \operatorname{Prim} A)$ *be the complete regularisation of* $\operatorname{Prim} A$*, as defined in Proposition 3.13, and let* $\Omega(A)$ *be the set of ideals*

$$\Omega(A) = \{\wedge\gamma(P) : P \in \operatorname{Prim} A\}.$$

Then the map $\gamma(P) \mapsto \wedge\gamma(P)$ is a bijection from $\gamma \operatorname{Prim} A$ onto $\Omega(A)$ and, for each P in $\operatorname{Prim} A$*,*

$$\gamma(P) = \operatorname{hull} \wedge\gamma(P).$$

Proof. The result follows from the observation that, for each P in $\operatorname{Prim} A$, $\gamma(P)$ is Jacobson closed in $\operatorname{Prim} A$. ■

Theorem 4.41 gives a more explicit realisation of the Stone-Čech compactification.

Theorem 4.41 *Let A be a JB*-triple and adopt the notation of Proposition 4.39. Identify $\gamma \operatorname{Prim} A$ with the set of ideals $\Omega(A)$ as in Lemma* 4.40*. For an element x*

of $\partial_e A_1^*$*, the set of pure functionals of* A*, let* J_x *be the norm central kernel of* x *in* A*, let* G_x *be the element* $\wedge\gamma(J_x)$ *of* $\Omega(A)$ *and let* $I_{\check{x}}$ *be the kernel in* $Z(A)$ *of the character* $\check{x}$ *induced on* $Z(A)$ *by* x *in Proposition 3.22. Then* $(\bigtriangledown, \mathrm{Prim}\, Z(A))$ *is a compactification of* $\Omega(A)$*, topologically equivalent to the Stone-Čech compactification of* $\Omega(A)$*, such that,*

$$G_x^{\bigtriangledown} = J_x^{\bigtriangledown} = I_{\check{x}}.$$

The inverse $I \mapsto I_{\triangle}$ *satisfies*

$$(I_{\check{x}})_{\triangle} = G_x.$$

Proof. By Proposition 4.39 and Lemma 4.36,

$$\begin{aligned}\gamma(J_x) &= \{P \in \mathrm{Prim}\, A : \gamma(P) = \gamma(J_x)\} \\ &= \{P \in \mathrm{Prim}\, A : \bigtriangledown' \circ \gamma(P) = \bigtriangledown' \circ \gamma(J_x)\} \\ &= \{P \in \mathrm{Prim}\, A : P^{\bigtriangledown} = I_{\check{x}}\}.\end{aligned}$$

Using the fact that $P^{\bigtriangledown}$ is a proper ideal of $Z(A)$, $I_{\check{x}}$ is a maximal ideal of $Z(A)$ and, by Lemma 4.37,

$$\begin{aligned}\gamma(J_x) &= \{P \in \mathrm{Prim}\, A : P^{\bigtriangledown} \in \mathrm{hull}\, I_{\check{x}}\} \\ &= \mathrm{hull}((I_{\check{x}})_{\triangle}).\end{aligned}$$

Taking intersections,

$$G_x = \cap\gamma(J_x) = (I_{\check{x}})_{\triangle}.$$

Using Lemma 4.35,

$$G_x^{\bigtriangledown} = (\cap\gamma(J_x))^{\bigtriangledown} = \cap\{P^{\bigtriangledown} : P \in \gamma(J_x)\}.$$

But it has just been shown that, for all P in $\gamma(J_x)$, $P^{\bigtriangledown}$ equals $I_{\check{x}}$. Thus

$$G_x^{\bigtriangledown} = I_{\check{x}},$$

as required. ■

Corollary 4.42 *Let A be a JB*-triple with centralizer $Z(A)$, let $\Omega(A)$ be the completely regular space of ideals of Theorem 4.41 and let $C^b(\Omega(A))$ be the space of bounded continuous complex-valued functions on $\Omega(A)$. Then there is an isometric *-isomorphism $T \mapsto g_T$ from $Z(A)$ onto $C^b(\Omega(A))$ such that, for all elements G in $\Omega(A)$ and a in A,*

$$Ta + G = g_T(G)a + G.$$

Proof. Let $\operatorname{Prim} Z(A)$ be the primitive spectrum of $Z(A)$. By Theorem 4.41, $(\triangledown, \operatorname{Prim} Z(A))$ is topologically equivalent to the Stone-Čech compactification of $\Omega(A)$. It follows that the map $h \mapsto h \circ \triangledown$ is an isometric *-isomorphism from $C(\operatorname{Prim} Z(A))$, the space of continuous complex-valued functions on $\operatorname{Prim} Z(A)$ onto $C^b(\Omega(A))$. By the Gelfand-Naimark theorem, there is an isometric *-isomorphism $T \mapsto h_T$ from $Z(A)$ onto $C(\operatorname{Prim} Z(A))$ such that, for I in $\operatorname{Prim} Z(A)$,

$$h_T(I)(Id_A + I) = T + I.$$

For T in $Z(A)$, let g_T be the function $G \mapsto h_T(G^{\triangledown})$. Then $T \mapsto g_T$ is an isometric *-isomorphism from $Z(A)$ onto $C^b(\Omega(A))$, such that, for G in $\Omega(A)$, $g_T(G)Id_A - T$ lies in $G^{\triangledown}$. Thus, for a in A, $g_T(G)a - Ta$ lies in $(G^{\triangledown})_{\triangle}$, which, by Theorem 4.41, coincides with G. ■

Chapter 5

Densely standard JB*-triples

In this chapter an investigation into the existence of a Gelfand representation for JB*-triples is described. Given a JB*-triple A, the task is to construct a locally compact Hausdorff space, Ω, associate to each point ω in Ω a JB*-triple A_ω, and represent each element a of A as an element of $\prod_{\omega\in\Omega} A_\omega$. The space Ω is said to be the base space, for ω in Ω the JB*-triple A_ω is said to be the fibre at ω, and the elements of $\prod_{\omega\in\Omega} A_\omega$ are said to be cross-sections. The aim is to develop a tool for reducing the study of a JB*-triple to the study of the more accessible sub-class of *primitive* JB*-triples, that is, the class of JB*-triples for which the zero ideal is primitive. To achieve this end, the construction will be subject to the conditions:

(S1) the range of the representation is a structure known as a maximal full triple of cross-sections, described in Section 5.2;

(S2) the set of elements ω in Ω for which the associated fibre A_ω is a primitive JB*-triple is dense in Ω.

Condition S1 ensures that, under the representation, the structure of A decomposes into the structure of the fibres in a natural way. Condition S2 ensures that, for a dense subset of Ω, the fibres can be embedded as w*-dense subsets of Cartan factors. A JB*-triple for which a construction satisfying conditions S2 and S1 exists is said to be *densely standard*. The aim of this chapter is to develop results to help establish which JB*-triples are densely standard.

In Section 5.1 the general theory of continuous cross-sections of Banach spaces over a locally compact Hausdorff space Ω is summarised, and specialised to the case in which the Banach spaces are JB*-triples. It is shown that, for a general JB*-triple A, if Ω is chosen to be the closure $\operatorname{Fell} A$ of the primitive spectrum $\operatorname{Prim} A$ of A in the Lawson topology, then there exists an isometric isomorphism, known as the Fell representation, from A into a JB*-triple of cross-sections over A.

In Section 5.2 the consequences of the JB*-triple A possessing a representation satisfying Condition S1 are investigated. Condition S1 ensures that the representation decomposes norm-closed inner ideals of A and pure functionals on A into the corresponding objects in the fibres, a result that leads to a Glimm Stone-Weierstrass theorem for A.

In Section 5.3 the question of identifying the base space Ω of a JB*-triple A known to possess a representation satisfying Condition S1 is addressed. It is shown that Ω can be constructed from $\operatorname{Prim} A$ by means of a Hausdorff equivalence relation. In the case in which Ω contains a dense set of proper primal ideals, the JB*-triple A is said to be *quasi-standard*. The main result of the chapter, Theorem 5.17, shows that if A is quasi-standard, then the base space Ω is uniquely determined up to homeomorphism as the space of minimal primal ideals in A with the relativised lower topology, which coincides with the relativised Lawson topology. Furthermore, the centralizer of A may be identified with the space $C^b(\Omega)$ of continuous bounded complex-valued functions on Ω. This result is significant because it applies to densely standard JB*-triples.

In Section 5.4 the Fell representation of a JB*-triple A is reconsidered. Since, by construction, the base space $\operatorname{Fell} A$ contains a dense set $\operatorname{Prim} A$ of primitive ideals, a natural question arises as to when the range is maximal. The results of the preceding sections are applied to show that the natural representation over the space $\operatorname{Fell}' A$ of proper Fell ideals is maximal if and only if $\operatorname{Prim} A$ is Hausdorff in the Jacobson topology. This is of course a restrictive condition, but the set of JB*-triples for which the condition holds includes JB*-triples of constant finite rank and abelian JB*-triples, thereby providing the first examples of densely standard JB*-triples.

The theory developed in this chapter will be applied in Chapter 6 to provide further examples of densely standard JB*-triples.

5.1 Full triples of cross-sections

In this section the machinery required to construct and study representations of JB*-triples is developed. Let Ω be a locally compact Hausdorff space and with each element ω in Ω associate a complex Banach space A_ω with norm $\|.\|_\omega$. The elements of the vector space $\prod_{\omega\in\Omega} A_\omega$ are said to be *cross-sections*, the space Ω is said to be the *base space*, and the spaces $\{A_\omega : \omega \in \Omega\}$ are said to be the *fibres*. Each cross-section a of $\prod_{\omega\in\Omega} A_\omega$ may be considered as a map on Ω such that, for each element ω in Ω, $a(\omega)$ lies in A_ω. A cross-section a is said to be *bounded* if the real-valued function $\rho_a : \omega \mapsto \|a(\omega)\|_\omega$ is bounded on Ω. For an element a in the subspace of bounded cross-sections, let

$$\|a\|_\infty = \sup_{\omega\in\Omega} \|a(\omega)\|_\omega.$$

Then $\|.\|_\infty$ is a norm on the space of bounded cross-sections. A vector subspace F of $\prod_{\omega\in\Omega} A_\omega$ such that:

(i) for each element b in F, the function $\rho_b : \omega \mapsto \|b(\omega)\|_\omega$ is continuous on Ω;

(ii) for each element ω in Ω, the set $\{b(\omega) : b \in F\}$ is dense in A_ω;

is said to be a *continuity structure* for $\prod_{\omega\in\Omega} A_\omega$. A cross-section a is said to be *continuous* with respect to F at an element ω_0 of Ω if, for any $\varepsilon > 0$, there exists an element b_ε in F and a neighbourhood U_ε of ω_0 such that, for all ω in U_ε,

$$\|(a - b_\varepsilon)(\omega)\|_\omega < \varepsilon.$$

The cross-section a is said to be F-continuous on Ω if it is F-continuous at all points of Ω. When each fibre A_ω coincides with a fixed Banach space A, a is continuous in the usual sense if and only if it is continuous with respect to the continuity structure of constant A-valued functions on Ω. For convenience, Lemma 5.1 records some elementary properties of continuity structures [42].

Lemma 5.1 *Let Ω be a locally compact Hausdorff space, and, for each element ω in Ω, let A_ω be a complex Banach space with norm $\|.\|_\omega$. Let F be a continuity structure for the space $\prod_{\omega\in\Omega} A_\omega$, let ω_0 be an element of Ω, and let $C_{\omega_0}(F)$ denote the set of cross-sections F-continuous at ω_0. Then, the following results hold.*

(i) *F is a subset of $C_{\omega_0}(F)$.*

(ii) *For each element a in $C_{\omega_0}(F)$ the function $\rho_a : \omega \mapsto \|a_\omega\|_\omega$ is continuous at ω_0.*

(iii) *$C_{\omega_0}(F)$ is a subspace of $\prod_{\omega\in\Omega} A_\omega$, closed under multiplication by the complex valued functions on Ω which are continuous at ω_0.*

(iv) *A cross-section a lies in $C_{\omega_0}(F)$ if and only if for each element b in F the function $\rho_{a-b} : \omega \mapsto \|(a-b)(\omega)\|_\omega$ is continuous at ω_0.*

(v) *Let $(a_n)_{n=1}^\infty$ be a sequence of cross-sections in $C_{\omega_0}(F)$, and let a be a cross-section such that, for each element n in $\mathbb{N}$, the sequence $(a-a_n)_{n=1}^\infty$ is bounded and the sequence $(\|a-a_n\|_\infty)_{n=1}^\infty$ converges to zero. Then a lies in $C_{\omega_0}(F)$.*

(vi) *For every element a_{ω_0} in A_{ω_0} there exists a F-continuous cross-section a such that $a(\omega_0)$ equals a_{ω_0}.*

By Lemma 5.1 (iii), the set of cross-sections F-continuous on Ω form a vector space, denoted by $C(\Omega, F)$. By Lemma 5.1 (v) and [56], Lemma 1.5.2, the subspace of bounded F-continuous cross-sections is a Banach space which we denote by $C^b(\Omega, F)$. An element a of $C^b(\Omega, F)$ is said to *vanish at infinity* if the function ρ_a vanishes at infinity. Since the space $C_0(\Omega)$ of complex-valued continuous functions on Ω vanishing at infinity is a closed subspace of $C^b(\Omega)$, it follows that $C_0(\Omega, F)$, the set of cross-sections vanishing at infinity, is a closed subspace of $C^b(\Omega, F)$ and, hence, a Banach space.

Now consider the situation in which, for each element ω in Ω, the fibre A_ω is a JB*-triple. Then, the pointwise defined triple product makes $\prod_{\omega\in\Omega} A_\omega$ into a Jordan *-triple. A continuity structure F is said to be a *triple continuity structure* if it is a subtriple of $\prod_{\omega\in\Omega} A_\omega$. It is straightforward to show that, if F is a triple continuity structure, then, for each element ω in Ω, the space $C_\omega(F)$ is a subtriple of $\prod_{\omega\in\Omega} A_\omega$. In this case, it follows that $C^b(\Omega, F)$ is a Jordan *-triple containing $C_0(\Omega, F)$ as a subtriple. In the situation considered by Fell [42] in which each fibre is in fact a C*-algebra, it is immediate that $C^b(\Omega, F)$ is also a C*-algebra. A new result, Lemma 5.2 below, is required to prove the equivalent statement for JB*-triples.

Let T be an element of a unital Banach algebra with numerical range $V(T)$. Define the real scalar $\mu(T)$ by

$$\mu(T) = \sup_{\lambda\in V(T)} \{\operatorname{Re}\lambda\} = \sup_{\alpha>0}\{\frac{1}{\alpha}\log\|\exp\alpha T\|\}.$$

Observe that

$$-\mu(-T) = \inf_{\lambda\in V(T)} \{\operatorname{Re}\lambda\}.$$

Since the numerical range of an hermitian operator is the convex hull of its spectrum, when T is hermitian $-\mu(-T)$ is positive if and only if T is positive. The reader is referred to [15] for details.

Lemma 5.2 *Let Ω be a locally compact Hausdorff space. For each element ω in Ω, let A_ω be a complex Banach space with norm $\|.\|_\omega$, let F be a continuity structure for the space $\prod_{\omega\in\Omega} A_\omega$ of cross-sections, and let $C^b(\Omega, F)$ be the Banach space of bounded F-continuous cross-sections on Ω. Let $(T_\omega)_{\omega\in\Omega}$ be a collection of maps such that:*

(i) *for each element ω in Ω, the map T_ω lies in $B(A_\omega)$, the Banach algebra of bounded linear operators on A_ω;*

(ii) *for each element a in $C^b(\Omega, F)$ the cross-section Ta defined, for ω in Ω, by $(Ta)(\omega) = T_\omega a(\omega)$ lies in $C(\Omega, F)$;*

(iii) *the set $\{\|T_\omega\| : \omega \in \Omega\}$ is bounded above.*

Then, the map $T : a \mapsto Ta$ is a bounded linear operator on $C^b(\Omega, F)$ such that,

$$\|T\| = \sup_{\omega\in\Omega} \|T_\omega\|,$$

$$\mu(T) \leq \sup_{\omega\in\Omega} \mu(T_\omega)$$

and, for all ω in Ω,

$$((\exp T)a)(\omega) = (\exp T_\omega)a(\omega).$$

Proof. The proof of the equalities is elementary. Let α be a positive scalar. Then

$$\|(\exp \alpha T)a\|_\infty \leq \sup_{\omega\in\Omega}\{\|\exp \alpha T_\omega\|_\omega\}\|a\|_\infty.$$

Since the logarithmic function is continuous and monotonic increasing,

$$\frac{1}{\alpha}\log\|(\exp \alpha T)\|_\infty \leq \sup_{\omega\in\Omega}\{\frac{1}{\alpha}\log\|\exp \alpha T_\omega\|_\omega\}.$$

Thus

$$\mu(T) \leq \sup_{\alpha>0}\sup_{\omega\in\Omega}\{\frac{1}{\alpha}\log\|\exp \alpha T_\omega\|_\omega\}.$$

Since α and ω are independent, the supremum operations commute to give the desired inequality. ■

Corollary 5.3 *Let Ω be a locally compact Hausdorff space and, for each element ω in Ω, let A_ω be a JB*-triple. Let F be a triple continuity structure for $\prod_{\omega\in\Omega} A_\omega$. Then $C^b(\Omega, F)$, the Banach space of bounded F-continuous cross-sections on Ω, is a JB*-triple, and $C_0(\Omega, F)$, the Banach subspace of F-continuous cross-sections vanishing at infinity, is a JB*-subtriple.*

Proof. Let a and b be elements of $C^b(\Omega, F)$. For ω in Ω, let $D(a(\omega))$ be the operator defined on A_ω by

$$D(a(\omega))b(\omega) = \{a\,a\,b\}(\omega).$$

For any complex number α, the collection of maps $(\alpha D(a(\omega)))_{\omega\in\Omega}$ satisfy the conditions of Lemma 5.2. Thus, a bounded linear operator $D(a)$ is defined on $C^b(\Omega, F)$ by

$$(D(a)b)(\omega) = D(a(\omega))b(\omega)$$

such that

$$\mu(\alpha D(a)) \leq \sup_{\omega\in\Omega}\mu(\alpha D(a(\omega))).$$

Putting α equal to i, $-i$ and -1,

$$\mu(\pm iD(a)) \leq \sup_{\omega\in\Omega} \mu(\pm iD(a(\omega))) = 0,$$

and

$$-\mu(-D(a)) \geq \inf_{\omega\in\Omega} -\mu(-D(a(\omega))) \geq 0.$$

It follows that $D(a)$ is positive and hermitian. Since

$$\|a^3\| = \sup_{\omega\in\Omega} \|a(\omega)\|^3 = \|a\|^3,$$

$C^b(\Omega, F)$ is a JB*-triple. Thus the closed subtriple $C_0(\Omega, F)$ is also a JB*-triple. ■

Proposition 5.4 *Let Ω be a locally compact Hausdorff space and, for each element ω in Ω, let A_ω be a JB*-triple. Let A be a triple continuity structure for $\prod_{\omega\in\Omega} A_\omega$ such that:*

(i) *for each element a in A, $\rho_a : \omega \mapsto \|a(\omega)\|$ vanishes at infinity;*

(ii) *A is complete in the supremum norm.*

Then A is a JB-subtriple of $C^b(\Omega, A)$, the JB*-triple of bounded A-continuous cross-sections, and for each ω in Ω,*

$$A_\omega = \{a(\omega) : a \in A\}.$$

Proof. Since A is a closed subspace of $C^b(\Omega, A)$, it is a JB*-triple. For ω in Ω and a in A, the map $a \mapsto a(\omega)$ defines a triple homomorphism from A onto a dense subset of A_ω. The result follows by [10], Lemma 1. ■

Given a locally compact space Ω and a JB*-triple A satisfying the conditions of Proposition 5.4, A of is said to be a *full triple of cross-sections* over Ω.

The next lemma shows that, subject to suitable conditions, sets of norm-closed ideals give rise to representations of JB*-triples onto full triples of cross sections in a natural way.

Lemma 5.5 *Let A be a JB*-triple and let Ω be a set of norm-closed ideals with trivial intersection, equipped with a locally compact Hausdorff topology such that, for each element a in A, the function $\rho_a : G \mapsto \|a + G\|$ is continuous and vanishes at infinity on Ω. To each element G in Ω, associate the fibre A/G. For each element a in A, let $\hat{a}$ be the cross section $\hat{a}$ of $\prod_{G\in\Omega} A/G$ defined by $\hat{a} : G \mapsto a + G$. Then $\hat{A}$, the set of such cross-sections, is a full triple of cross-sections over Ω, and the map $a \mapsto \hat{a}$ is an isometric isomorphism of A onto $\hat{A}$.*

Proof. That $\hat{A}$ is complete in the supremum norm follows from Lemma 4.26. The result is then immediate from the hypothesis. ■

When the conditions of Lemma 5.5 are satisfied, the representation $a \mapsto \hat{a}$ is said to be the *quotient representation* over Ω. As a first example of a quotient representation of a JB*-triple A onto a full triple of cross-sections, consider the base space Fell A, defined to be the closure of the primitive spectrum Prim A in the Lawson topology. By Proposition 4.32, Fell A satisfies the conditions of Lemma 5.5. By construction, Fell A possesses a dense subset of primitive ideals, and hence satisfies condition $S2$ of the introduction. However, it will emerge in Theorem 5.21 that this representation only satisfies condition $S1$, the important property of maximality, for a sub-class of the JB*-triples.

5.2 Maximal full triples of cross-sections

A full triple of cross-sections A over a locally compact Hausdorff space Ω is said to be *maximal* if, whenever B is a full triple of cross-sections over Ω such that A is a subspace of B, then A and B coincide. The interest in full triples of cross-sections which are maximal stems from the fact, established in this section, that the structure of a maximal full triple of cross-sections decomposes into the structure of its fibres. This leads to a Glimm Stone-Weierstrass theorem for maximal full triples of cross-sections.

A standard tool in the proofs of this section will be partitions of unity. Recall that, given a locally compact Hausdorff space Ω, a compact subset K of Ω, and an open cover $U_1, \ldots, U_n$ of K, there exist functions $f_1, \ldots, f_n$ such that, for each j in $1, \ldots, n$, f_j has compact support contained in U_j, f_j takes values in the range $[0, 1]$ and $\Sigma_{j=1}^n f_j$ coincides with 1 on K. Such a collection $f_1, \ldots, f_n$ is said to be a *partition of unity* on K, subordinate to the cover $U_1, \ldots, U_n$.

The discussion begins by establishing some equivalent conditions for maximality.

Lemma 5.6 *Let A be a full triple of cross-sections on the locally compact Hausdorff space Ω, let $C^b(\Omega)$ be the space of continuous bounded complex-valued functions on Ω and let $C_0(\Omega, A)$ be the space of A-continuous cross-sections vanishing at infinity. Then, the following conditions are equivalent:*

(i) *A is a maximal full triple of cross-sections;*

(ii) *A coincides with $C_0(\Omega, A)$;*

(iii) *there exists a triple continuity structure F such that A coincides with the space $C_0(\Omega, F)$ of F-continuous cross-sections vanishing at infinity;*

(iv) *A is closed under multiplication by elements of $C^b(\Omega)$.*

Proof. Given any triple continuity structure F, Lemma 5.1 and Urysohn's Lemma show that $C_0(\Omega, F)$ is a full triple of cross-sections. The implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii) are then obvious. The implication (iii)$\Rightarrow$(iv) follows from Lemma 5.1 (iii).

(iv)$\Rightarrow$(ii) Suppose that A is closed under multiplication by elements of $C^b(\Omega)$. Let a be an element of $C_0(\Omega, A)$ and let $\varepsilon > 0$. For each element ω in Ω there exists an element b_ω in A and an open neighbourhood U_ω of ω such that $\|(a - b_\omega)(\omega')\| < \varepsilon$ for all ω' in U_ω. Let K denote the compact set $\rho_a^{-1}([\varepsilon, \infty))$. The family $\{U_\omega : \omega \in K\}$ forms an open cover for K. By compactness there exist $\omega_1, \ldots, \omega_n$ in K such that $U_{\omega_1}, \ldots, U_{\omega_n}$ is an open cover of K. Let $f_1, \ldots, f_n$ be a partition of unity on K subordinate to this open cover Ω, and let f_0 be the function $1_\Omega - \sum_{j=1}^n f_j$. Then, for any element ω of Ω,

$$\|(a - \sum_{j=1}^{n} f_j b_{\omega_j})(\omega)\|_\omega \leq f_0(\omega)\|a(\omega)\|_\omega + \sum_{j=1}^{n} f_j(\omega)\|(a - b_{\omega_j})(\omega)\|_\omega.$$

For $j = 1, \ldots, n$, if ω lies in U_j then $\|(a - b_{\omega_j})(\omega)\| < \varepsilon$, and, otherwise $f_j(\omega)$ is zero. If ω lies in K, $f_0(\omega)$ is zero, and otherwise $\|a(\omega)\|_\omega < \varepsilon$. Thus, a lies in A and (ii) holds as required.

(ii)$\Rightarrow$(i) Suppose that A coincides with $C_0(\Omega, A)$. Let B be a full triple of cross sections over Ω containing A. Let b be an element of B. Then, for any element a of A, $b-a$ lies in B and hence $\omega \mapsto \|(b-a)(\omega)\|$ is continuous on Ω. By Lemma 5.1(iv), it follows that b is A-continuous. Then b lies in $C_0(\Omega, A)$ and hence in A. It follows that A is maximal. ■

Let A be a maximal full triple of cross-sections on a locally compact Hausdorff space Ω. For ω in Ω, let A_ω denote the JB*-triple

$$A_\omega = \{a(\omega) : a \in A\}.$$

A subtriple B of A is said to *separate points* in $\sqcup_{\omega\in\Omega} A_\omega$ if,

(i) for any elements ω in Ω and α in A_ω, there exists an element b in B such that b takes the value α at ω,

(ii) for any distinct elements ω_1 and ω_2 of Ω and any elements α_1, α_2 of A_{ω_1} and A_{ω_2} respectively, there exists an element b in B such that

$$b(\omega_1) = \alpha_1, \qquad b(\omega_2) = \alpha_2.$$

Note that when Ω contains more than one element, (ii) implies (i).

Lemma 5.7 *Let A be a maximal full triple of cross-sections on a locally compact Hausdorff space Ω. Then A separates points in $\sqcup_{\omega\in\Omega} A_\omega$.*

Proof. Let ω_1 and ω_2 be distinct points of Ω, let α_1 be an element of A_{ω_1}, and let α_2 be an element of A_{ω_2}. By Urysohn's lemma there exists elements a_1 and a_2 in $C_0(\Omega, F)$ with disjoint compact supports such that $a_1(\omega_1)$ equals α_1 and $a_2(\omega_2)$ equals α_2. Letting a be the element $a_1 + a_2$ gives the desired result. ■

This section will conclude with a Stone-Weierstrass theorem for maximal full triples of cross-sections (Theorem 5.13) which forms a converse to Lemma 5.7. Before this converse can be proved, it is necessary to describe the structure of a maximal full triple of cross-sections in terms of the structure of its fibres. This description begins with a study of inner ideals.

Lemma 5.8 *Let A be a maximal full triple of cross-sections over the locally compact Hausdorff space Ω and let I be a norm-closed inner ideal of A. Then I is closed under multiplication by elements of $C^b(\Omega)$, the space of bounded complex-valued continuous functions on Ω.*

Proof. For an element a in A, let ρ_a be the mapping $\omega \mapsto \|a(\omega)\|$. Let b be an element of I and let ω be an element of Ω. By Proposition 2.22, $\overline{\{b(\omega)\, A_\omega\, b(\omega)\}}^n$ is the smallest norm-closed inner ideal of A_ω containing $b(\omega)$. Thus, for any $\varepsilon > 0$, there exists an element a_ω in A such that

$$\|b(\omega) - \{b(\omega)\, a_\omega(\omega)\, b(\omega)\}\|_\omega < \varepsilon.$$

Let U_ω denote the open neighbourhood $\rho^{-1}_{b-\{b\, a_\omega\, b\}}(-\varepsilon, \varepsilon)$ of ω, and let K denote the compact set $\rho_b^{-1}([\varepsilon, \infty))$. By compactness, there exist elements $\omega_1, \ldots, \omega_n$ in K such that $\{U_{\omega_1}, \ldots, U_{\omega_n}\}$ forms an open cover of K. Let $f_1, \ldots, f_n$ be a partition of unity on K subordinate to this open cover Ω and let f_0 be the function $1_\Omega - \sum_{j=1}^n f_j$. Then, for any element ω of Ω,

$$\begin{aligned}\|(b - \textstyle\sum_{j=1}^n \{b\, f_j a_{\omega_j}\, b\})(\omega)\|_\omega &\leq \|f_0(\omega)\|\|b(\omega)\|_\omega + \sum_{j=1}^n \|f_j(\omega)\|_\omega \|(b - \{b\, a_{\omega_j}\, b\})(\omega)\|_\omega \\ &< \varepsilon.\end{aligned}$$

Let g be any element of $C^b(\Omega)$. Then,

$$\|gb - \textstyle\sum_{j=1}^n \{b\, f_j \bar{g} a_{\omega_j}\, b\}\|_\infty < \varepsilon \|g\|_\infty$$

Since $\{b\, \sum_{j=1}^n f_j \bar{g} a_{\omega_j}\, b\}$ is an element of I and I is norm-closed, it follows that gb

must also lie in I. ■

Proposition 5.9 *Let A be a JB*-triple which is a maximal full triple of cross-sections over a locally compact Hausdorff space Ω and let I be a norm-closed inner ideal of A. For each element ω in Ω, the set I_ω defined by*

$$I_\omega = \{a(\omega) : a \in I\}$$

is a norm-closed inner ideal of A_ω and

$$I = \{a \in A : a(\omega) \in I_\omega \quad \forall \omega \in \Omega\}.$$

Proof. Let a be an element of A such that $a(\omega)$ lies in I_ω for all ω in Ω and let $\varepsilon > 0$. For ω_0 in Ω there exists b_{ω_0} in I such that a and b_{ω_0} agree at ω_0. By continuity of $\rho_{a-b_{\omega_0}}$, there exists an open neighbourhood U_{ω_0} of ω_0 such that $\|(a - b_{\omega_0})(\omega)\| < \varepsilon$ for all ω in U_{ω_0}. By a standard partition of unity argument,

$$\|a - \sum_{j=1}^{n} f_j b_{\omega_j}\|_\infty < \varepsilon$$

for some $\omega_1, \ldots, \omega_n$ in Ω and some partition of unity $f_0, f_1, \ldots, f_n$ on K subordinate to the open cover $\Omega \backslash K, U_{\omega_1}, \ldots, U_{\omega_n}$ of Ω. Thus a lies in I.

Since I is a norm-closed subset of $C^b(\Omega, A)$, the space of A-continuous cross-sections on Ω, and closed under multiplication by elements of $C^b(\Omega)$, the space of continuous bounded complex valued functions on Ω, it follows that I_ω is norm-closed in A_ω. Clearly I_ω is an inner ideal of A_ω. ■

Let A be a JB*-triple. Recall, from Section 3.3, that the elements of $\partial_e A_1^*$, the set of extreme points of the dual unit ball A_1^* of A are said to be the pure functionals of A. It is now possible to describe the pure functionals of a full triple of cross-sections. The result could alternatively have been expressed in terms of Cartan factor representations, which would reveal it to be an extension of [42], Theorem 1.1 to the JB*-triple setting.

Theorem 5.10 *Let A be a JB*-triple that is a full triple of cross-sections over a locally compact Hausdorff space Ω, let A_ω denote the fibre at the element ω in Ω, and let $\partial_e A_1^*$ and $\partial_e(A_\omega)_1^*$ denote respectively the set of pure functionals of A and A_ω. Let $\sqcup_{\omega\in\Omega}\partial_e(A_\omega)_1^*$ be the set*

$$\sqcup_{\omega\in\Omega}\partial_e(A_\omega)_1^* = \{(\omega, x) : \omega \in \Omega, x \in \partial_e(A_\omega)_1^*\}.$$

For each pair (ω, x) in $\sqcup_{\omega\in\Omega}\partial_e(A_\omega)_1^*$, define a functional $\Phi_{\omega,x}$ for a in A by

$$\Phi_{\omega,x}(a) = x(a(\omega)).$$

Then the map $\Phi : (\omega, x) \mapsto \Phi_{\omega,x}$ maps $\sqcup_{\omega\in\Omega}\partial_e(A_\omega)_1^*$ into $\partial_e A_1^*$. If A is maximal then Φ is a bijection.

Proof. For each element ω in Ω, let $\delta_\omega : A \mapsto A_\omega$ denote the surjective triple homomorphism $a \mapsto a(\omega)$ and let $\tilde{\delta_\omega} : A/(\ker\delta_\omega) \mapsto A_\omega$ be the corresponding triple isomorphism. By Theorem 2.13, $\tilde{\delta_\omega}$ is an isometric triple isomorphism and induces an isometric isomorphism $\phi_\omega : A_\omega^* \mapsto (\ker\delta_\omega)^\circ$ such that, for x in A_ω^* and a in A,

$$(\phi_\omega x)(a) = x(\tilde{\delta_\omega}(a + \ker\delta_\omega)) = x(a(\omega)).$$

Thus $\Phi : (\omega, x) \mapsto \phi_\omega x$ is a well-defined map from $\sqcup_{\omega\in\Omega}\partial_e(A_\omega)_1^*$ into $\partial_e A_1^*$.

Now consider the case when A is maximal. Let y be an element of $\partial_e A_1^*$ and let I be the norm central kernel of y. For ω in Ω, let I_ω be the ideal of A_ω,

$$I_\omega = \{a(\omega) : a \in I\},$$

and let Λ be the set

$$\Lambda = \{\omega \in \Omega : I_\omega \neq A_\omega\}.$$

Since y is non-zero, by Proposition 5.9, Λ is non-empty. Assume that Λ contains distinct elements ω_1 and ω_2. Since Ω is Hausdorff, there exists disjoint neighbourhoods U_1 and U_2 of ω_1 and ω_2 respectively. Clearly, for j equal to $1, 2$, the set K_j of elements of A vanishing outside of U_j is an ideal of A and K_1 and K_2 have zero intersection. Since I is primitive, it is prime, and, interchanging ω_1 and ω_2 if necessary, it may be assumed that K_1 is a subset of I. This implies that

$$\{a(\omega_1) : a \in K_1\} \subseteq I_{\omega_1} \subseteq A_{\omega_1}.$$

But, by Urysohn's Lemma, every element of A_{ω_1} is of the form $a(\omega_1)$ for some a in K_1. Thus I_{ω_1} and A_{ω_1} coincide, contradicting the choice of ω_1 as an element of Λ. Thus Λ contains a single point ω_y. By Proposition 5.9,

$$I = \{a \in A : a(\omega_y) \in I_{\omega_y}\}.$$

Observe that $\ker \delta_{\omega_y}$ is a subset of I, and, hence, y lies in $\partial_e(\ker \delta_{\omega_y})_1^\circ$, the set of extreme points of the unit ball of $(\ker \delta_{\omega_y})^\circ$. Let x_y be the element $\phi_{\omega_y}^{-1} y$ of $\partial_e(A_{\omega_y})_1^*$. Then,

$$\Phi(\omega_y, x_y) = \phi_{\omega_y} x_y = y.$$

and Φ is surjective. Let (ω, x) be a pair in $\sqcup_{\omega \in \Omega} \partial_e(A_\omega)_1^*$ such that y equals $\Phi(\omega, x)$. Assume that ω is distinct from ω_y. Then ω does not lie in Λ and hence, for any α in a_ω, there exists a in I such that

$$x(\alpha) = x(a(\omega)) = y(a) = 0,$$

contradicting the fact that x has norm 1. Thus ω coincides with ω_y and

$$x = \phi_{\omega_y}^{-1} y = x_y.$$

This shows that ϕ is injective. ■

Corollary 5.11 *Let A be a JB*-triple that is a full triple of cross-sections over a locally compact Hausdorff space Ω, let A_ω denote the fibre at the element ω in Ω and let $C_0(\Omega, A)$ denote the space of A-continuous cross-sections vanishing at infinity.*

Let $\partial_e C_0(\Omega, A)_1^*$, $\partial_e A_1^*$ and $\partial_e (A_\omega)_1^*$ denote respectively the set of pure functionals of $C_0(\Omega, A)$, A and A_ω. Let $\sqcup_{\omega \in \Omega} \partial_e (A_\omega)_1^*$ be the set

$$\sqcup_{\omega \in \Omega} \partial_e (A_\omega)_1^* = \{(\omega, x) : \omega \in \Omega, x \in \partial_e (A_\omega)_1^*\}.$$

For each pair (ω, x) in $\sqcup_{\omega \in \Omega} \partial_e (A_\omega)_1^*$, define a functional $\Phi_{\omega,x}$ for a in $C_0(\Omega, A)$ by

$$\Phi_{\omega,x}(a) = x(a(\omega))$$

Then the map $\Phi : (\omega, x) \mapsto \Phi_{\omega,x}$ is a bijection from $\sqcup_{\omega \in \Omega} \partial_e (A_\omega)_1^*$ onto $\partial_e C_0(\Omega, A)_1^*$ and the map $\Phi : (\omega, x) \mapsto \Phi_{\omega,x}|_A$ is a surjection from $\sqcup_{\omega \in \Omega} \partial_e (A_\omega)_1^*$ onto $\partial_e A_1^*$.

Proof. By Lemma 5.6, $C_0(\Omega, A)$ is maximal. Applying Theorem 5.10 to both $C_0(\Omega, A)$ and A shows that Φ is a bijection from $\sqcup_{\omega \in \Omega} \partial_e (A_\omega)_1^*$ onto $\partial_e C_0(\Omega, A)_1^*$ and the range of the map $\Phi : (\omega, x) \mapsto \Phi_{\omega,x}|_A$ is a subset of $\partial_e A_1^*$. Let z be an element of $\partial_e A_1^*$. By the Hahn-Banach and Krein-Milman Theorems, there exists an element y in $\partial_e C_0(\Omega, A)_1^*$ extending z. Since Φ is a bijection, there exist ω in Ω and x in $\partial_e (A_\omega)_1^*$ such that y equals $\Phi(\omega, x)$. Thus z equals $\Phi_{\omega,x}|_A$ and the map $x \mapsto \Phi_{\omega,x}|_A$ is surjective. ■

Corollary 5.12 *Let A be a JB*-triple that is a full triple of cross-sections over a locally compact Hausdorff space Ω and let A_ω denote the fibre at the element ω of Ω. Let $\overline{\partial_e A_1^*}^{w*}$ be the w*-closure of the set of pure functionals of A and let $(A_\omega)_1^*$ be the dual unit ball of A_ω. Then, for every element x in $\overline{\partial_e A_1^*}^{w*} \cup \{0\}$, there exists an element ω in Ω and an element y in $(A_\omega)_1^*$ such that, for all a in A*

$$x(a) = y(a(\omega)).$$

Proof. When x is equal to zero, the result is trivial. If x is non-zero then there exists a net (x_λ) in $\partial_e A_1^*$ which is w*-convergent to x. By Corollary 5.11, there exists a net (ω_λ) in Ω and, for each element λ, an element y_λ in $\partial_e (A_{\omega_\lambda})_1^*$, the set of pure functionals of A_{ω_λ}, such that, for all elements a in A,

$$x_\lambda(a) = y_\lambda(a(\omega_\lambda)).$$

Choose a in A such that $x(a)$ is non-zero, and let ε be equal to $1/2|x(a)|$. Then, there exists an element λ_0 such that, for all $\lambda \geq \lambda_0$,

$$2\varepsilon = |x(a)| \leq |x_\lambda(a) - x(a)| + |x_\lambda(a)| < \varepsilon + |x_\lambda(a)|.$$

Thus, for all $\lambda \geq \lambda_0$,

$$\varepsilon < |y_\lambda(a(\omega_\lambda))| \leq \|a(\omega_\lambda)\|_{\omega_\lambda}. \tag{5.2.1}$$

Since the map $\rho_a : \omega \mapsto \|a(\omega)\|$ vanishes at infinity, $(\omega_\lambda)_{\lambda \geq \lambda_0}$ is contained in a compact subset of Ω and therefore has a subnet (ω_μ) converging to a point ω in Ω. Let $\delta_\omega : A \mapsto A_\omega$ be the surjective triple homomorphism $a \mapsto a(\omega)$ Taking limits in 5.2.1 shows that a does not lie in $\ker \delta_\omega$. Thus, x lies in $(\ker \delta_\omega)^\circ_1$ and we can define y to be equal to $(\delta^*_\omega)^{-1}x$. ■

The concluding result of this section is the converse to Lemma 5.7, a Glimm Stone-Weierstrass theorem for maximal full triples of cross-sections.

Theorem 5.13 *Let A be a JB*-triple that is a maximal full triple of cross-sections over a locally compact Hausdorff space Ω and let B be a norm-closed subtriple of A. Then B coincides with A if and only if B separates points in $\sqcup_{\omega\in\Omega} A_\omega$.*

Proof. Let A be maximal and let B separate A. For ω in Ω, let A_ω be the fibre associated with ω. By the Glimm Stone-Weierstrass Theorem for JB*-triples [68], it is sufficient to show that B separates points in $\overline{\partial_e A^*_1}^{w*} \cup \{0\}$. To this end, let x_1 and x_2 be distinct elements of $\overline{\partial_e A^*_1}^{w*} \cup \{0\}$. By Corollary 5.12, there exists elements ω_1 and ω_2 in Ω and y_1 in $(A_{\omega_1})^*_1$ and y_2 in $(A_{\omega_2})^*_1$ such that, for all elements a in A

$$x_1(a) = y_1(a(\omega_1)), \qquad x_2(a) = y_2(a(\omega_2))$$

Since x_1 and x_2 are distinct, there exists an element a in A separating x_1 and x_2. By hypothesis, there exists an element b in B such that,

$$b(\omega_1) = a(\omega_1), \qquad b(\omega_2) = a(\omega_2)$$

Therefore,

$$x_1(b) = y_1(b(\omega_1)) = y_1(a(\omega_1)) = x_1(a),$$
$$x_2(b) = y_2(b(\omega_2)) = y_2(a(\omega_2)) = x_2(a).$$

Thus the element b separates x_1 from x_2. ∎

When A is a C*-algebra, the results presented here extend those obtained in [42] for norm-closed left ideals to norm-closed inner ideals and from pure states to pure functionals.

5.3 Quasi-standard JB*-triples

This section contains the main results of the chapter. First, it is shown in Theorem 5.14 that, under favourable circumstances, namely the existence of an open Hausdorff equivalence relation on the primitive spectrum, it is possible to construct a representation of a JB*-triple onto a maximal full triple of cross sections over a base space of norm-closed ideals. Theorem 5.16 shows that, conversely, if a JB*-triple possesses a representation onto a maximal full triple of cross sections over a base space of norm-closed ideals, then the primitive spectrum possesses an open Hausdorff equivalence relation and the representation arises from the construction of Theorem 5.14. This then leads to the main result of the chapter, Theorem 5.17, in which it is shown that the JB*-triple A is quasi-standard if and only if the canonical representation of A as a full triple of cross-sections over the set of minimal primal ideals is an isometric *-isomorphism onto a maximal full triple of cross-sections.

Proposition 3.13, the construction of the complete regularisation, gives an example of a Hausdorff equivalence relation on the primitive spectrum of any JB*-triple.

Theorem 5.14 *Let A be a JB*-triple, let* $\operatorname{Prim} A$ *be the primitive spectrum of A, equipped with the Jacobson topology, let $\sim$ be a Hausdorff equivalence relation on* $\operatorname{Prim} A$*, and let* $\operatorname{Prim} A/\sim$ *be the set of equivalence classes of elements of* $\operatorname{Prim} A$*. For P in* $\operatorname{Prim} A$*, let $[P]$ be the equivalence class containing P, let $G_{[P]}$ be the norm-closed ideal*

$$G_{[P]} = \wedge\{Q : Q \in [P]\}.$$

and let Ω be the space

$$\Omega = \{G_{[P]} : P \in \operatorname{Prim} A\}$$

equipped with the quotient topology $\mathfrak{Q}$ induced by the map $P \mapsto G_{[P]}$. For each element G in Ω, let the fibre A_G be the JB-triple A/G, and for each element a in A and G in Ω, let $\hat{a}$ be the cross-section of $\prod_{G\in\Omega} A_G$ defined for G in Ω by*

$$\hat{a}(G) = a + G.$$

Then, the following results hold.

(i) *The mapping $[P] \mapsto G_{[P]}$ is a homeomorphism from* $\operatorname{Prim} A/\sim$ *in the quotient topology onto Ω.*

(ii) *For each element a in A, the mapping $\rho_a : G \mapsto \|a+G\|$ is upper semi-continuous on Ω, with respect to any Hausdorff topology weaker than $\mathfrak{Q}$.*

(iii) *The following conditions are equivalent:*

(a) the equivalence relation $\sim$ is open;

(b) for all elements a in A, ρ_a is lower semi-continuous on Ω;

(c) Ω is locally compact and for all elements a in A, ρ_a is lower semi-continuous on Ω;

(d) Ω is locally compact, $\hat{A}$, the set $\{\hat{a} : a \in A\}$, is a maximal full triple of cross-sections over Ω and the map $a \mapsto \hat{a}$, is an isometric isomorphism

from A onto $\hat{A}$.

Proof. (i) Since $\operatorname{Prim} A/\sim$ is Hausdorff in the quotient topology, the equivalence classes are closed sets in $\operatorname{Prim} A$, and the map $[P] \mapsto G_{[P]}$ has inverse $G_{[P]} \mapsto \operatorname{hull} G_{[P]}$. By construction, $[P] \mapsto G_{[P]}$ is a homeomorphism.

(ii) Let $\mathfrak{T}$ be a Hausdorff topology on Ω, weaker than $\mathfrak{Q}$. For an element a in A and $\alpha \geq 0$, let G be an element of Ω such that $\|a+G\| \geq \alpha$. By Lemma 4.27, there exists a primitive ideal Q in $\operatorname{hull} G$ such that

$$\|a+Q\| = \|a+G\| \geq \alpha.$$

Let $q : \operatorname{Prim} A \mapsto \Omega$ be the $\mathfrak{T}$-continuous quotient map $P \mapsto G_{[P]}$. Then

$$\rho_a^{-1}([\alpha,\infty)) = \{G \in \Omega : \|a+G\| \geq \alpha\} = q(\{P \in \operatorname{Prim} A : \|a+P\| \geq \alpha\})$$

and since $\mathfrak{T}$ is Hausdorff, as the continuous image of a compact set (Proposition 4.28), $\rho_a^{-1}([\alpha,\infty))$ is closed. It follows that ρ_a is $\mathfrak{T}$ upper semi-continuous.

(iii) (a)$\Leftrightarrow$(b) By the argument above, for an element a in A and $\alpha \geq 0$,

$$q(\{P \in \operatorname{Prim} A : \|a+P\| > \alpha\}) = \{G \in \Omega : \|a+G\| > \alpha\}$$

The equivalence of the conditions now follows from Lemma 4.30.

(b)$\Rightarrow$(c) By (a), the map q is an open continuous surjection. Since $\operatorname{Prim} A$ is locally compact (Proposition 4.31) Ω is locally compact.

(c)$\Rightarrow$(d) It follows from (ii) and (c) that for all a in A, ρ_a is continuous, and by Lemma 5.5, $\hat{A}$ is a full triple of cross-sections. Let B be the maximal full triple of operator fields $C_0(\Omega, \hat{A})$. By Theorem 5.13, in order to show that $\hat{A}$ and B coincide, it is sufficient to show that for distinct elements J_1 and J_2 in Ω and elements a_1 and a_2 in A, there exists a in A such that,

$$a + J_1 = a_1 + J_1, \qquad a + J_2 = a_2 + J_2. \tag{5.3.1}$$

Assume that the norm-closed ideal $J_1 + J_2$ is proper. Let P be a primitive ideal containing $J_1 + J_2$ and let Q_1, Q_2 be primitive ideals such that the equivalence classes $[Q_1]_\sim$ and $[Q_2]_\sim$ are equal to $\operatorname{hull} J_1$ and $\operatorname{hull} J_2$ respectively. Then P lies in $[Q_1]_\sim \cap [Q_2]_\sim$, contradicting the assumption that J_1 and J_2 are distinct. Thus $J_1 + J_2$ is equal to A and there exist elements b_1 in J_1 and b_2 in J_2 such that

$$a_1 - a_2 = b_1 - b_2.$$

Define a in A by

$$a = a_1 - b_1 = a_2 - b_2.$$

Then a satisfies equation 5.3.1 as required.

(d)⇒(b) By definition, for all elements a in A, ρ_a is continuous and, in particular, lower semi-continuous. ■

Corollary 5.15 is recorded for reference in Chapter 6.

Corollary 5.15 *Let A be a JB*-triple and let $\Omega(A)$ be the complete regularisation of the primitive spectrum of A, identified with a set of ideals of A as in Lemma 4.40. For each element a in A, the mapping $\rho_a : G \mapsto \|a + G\|$ is upper semi-continuous on $\Omega(A)$. The restriction of the Scott topology to $\Omega(A)$ is weaker than the completely regular topology on $\Omega(A)$.*

Proof. The completely regular topology is a Hausdorff topology weaker than the quotient topology on $\Omega(A)$. The result is therefore immediate from Theorem 5.14 and Proposition 4.32. ■

Theorem 5.16 *Let A be a JB*-triple possessing a set Ω of norm-closed ideals equipped with a locally compact Hausdorff topology $\mathfrak{T}$ such that the quotient representation of A over Ω is an isometric *-isomorphism onto a maximal full triple of cross-sections over Ω. Let* $\operatorname{Prim} A$ *be the primitive spectrum of A equipped with the Jacobson topology and let the relation $\sim$ on* $\operatorname{Prim} A$ *be defined for primitive ideals P*

and Q by $P \sim Q$ if and only if $P \cap Q$ contains an element I of Ω. Then $\sim$ is an open equivalence relation on $\mathrm{Prim}\, A$ *with quotient space*

$$\mathrm{Prim}\, A/\sim\, = \{\mathrm{hull}\, I : I \in \Omega\},$$

and the map $p : S \mapsto \wedge S$ is a homeomorphism from $\mathrm{Prim}\, A/\sim$ *equipped with the quotient topology onto Ω.*

Proof. Let P be a primitive ideal of A. Then P is the norm central kernel $k_n(\ker x)$ of some pure functional x of A. By Theorem 5.10, there exist elements I in Ω and y in $\partial_e(A/I)_1^*$ such that, for all elements a in A,

$$x(a) = y(a + I).$$

Therefore, P contains the element I of Ω. If J is also an element of Ω contained in P then x lies in $\partial_e(A/J)_1^*$ and by Theorem 5.10, J coincides with I. Hence, each primitive ideal contains a unique element of Ω. It follows that $\sim$ is an equivalence relation and that the equivalence classes are exactly the sets $\mathrm{hull}\, I$ for I in Ω. It is now immediate from Proposition 3.10 that p is a bijection onto Ω. Let q be the quotient map $q : \mathrm{Prim}\, A \mapsto \mathrm{Prim}\, A/\sim$. The argument of [70], Theorem 1.3 shows that $q \circ p$ is continuous and the argument of [61], Theorem 4 shows that $q \circ p$ is open. It is then elementary that p is a homeomorphism and that q is open. ■

A JB*-triple is said to be *quasi-standard* if it possesses a base space Ω of norm-closed ideals, equipped with a locally compact Hausdorff topology $\mathfrak{T}$ such that Ω possesses a $\mathfrak{T}$-dense set of proper primal ideals and A is isometrically *-isomorphic to a maximal full triple of cross-sections over Ω.

Theorem 5.17, the critical result of this chapter, describes the particularly satisfactory representation theory of quasi-standard JB*-triples. This result will be useful in the sequel for identifying classes of JB*-triples which are densely-standard JB*-triples.

Theorem 5.17 *Let A be a quasi-standard JB*-triple over a base space of norm-closed ideals Ω with locally compact Hausdorff topology $\mathfrak{T}$. Let $\operatorname{Prim} A$ be the primitive spectrum of A and identify its complete regularisation $\gamma \operatorname{Prim} A$ with the set of ideals $\Omega(A)$ as in Lemma 4.40. The the following topological spaces coincide as sets and are homeomorphic under the identity map:*

(i) Ω *in the $\mathfrak{T}$ topology;*

(ii) $\operatorname{MinPrimal} A$ *in the relative lower topology;*

(iii) $\operatorname{MinPrimal} A$ *in the relative Lawson topology;*

(iv) $\Omega(A)$ *in the completely regular topology induced from $\gamma \operatorname{Prim} A$.*

Proof. It follows from Lemma 4.33 that $\mathfrak{T}$ is stronger than the restriction of the Lawson and lower topologies to Ω. Since A is quasi-standard, this implies that every element of Ω is a limit in the lower topology of primal ideals and therefore primal by Theorem 3.11. By Theorem 5.16, an open equivalence relation $\sim$ is defined on $\operatorname{Prim} A$ by $P \sim Q$ if and only if $P \cap Q$ contains an element I of Ω. Let $\approx$ be the equivalence relation defined on $\operatorname{Prim} A$ in Proposition 3.13. As in [7], Theorem 3.4, for elements P and Q in $\operatorname{Prim} A$, $P \sim Q$ if and only if, for all continuous complex-valued bounded functions f on $\operatorname{Prim} A$, f takes the same value at P as at Q, if and only if $P \approx Q$. Thus $\gamma \operatorname{Prim} A$ is the set of $\sim$-equivalence classes and Ω and $\Omega(A)$ coincide as sets. By Theorem 3.17,

$$\Omega = \Omega(A) = \operatorname{MinPrimal} A.$$

By Theorem 5.16, the map $p : S \mapsto \wedge S$ is a homeomorphism from $\gamma \operatorname{Prim} A$ with the quotient topology to Ω with the $\mathfrak{T}$ topology. Applying Proposition 3.15, p is also a homeomorphism from $\gamma \operatorname{Prim} A$ with the quotient topology onto Ω equipped with the lower topology. It follows that $\mathfrak{T}$ coincides with the restriction of the lower topology to Ω. By [53], Theorem 6.6 (b), the Lawson topology coincides with the lower topology on Ω. By Theorem 5.14, the quotient topology on $\gamma \operatorname{Prim} A$ is locally compact and Hausdorff, and hence completely regular. By Proposition 3.14, the identity map on

$\gamma \operatorname{Prim} A$ is continuous from the completely regular topology to the quotient topology. Thus the quotient and completely regular topologies coincide on $\gamma \operatorname{Prim} A$. The result follows. ■

Corollary 5.18, an easy consequence of Theorem 5.17 and Corollary 4.42, shows that the action of the centralizer on a quasi-standard JB*-triple decomposes into pointwise multiplication by a continuous bounded function over the base space.

Corollary 5.18 *Let A be a quasi-standard JB*-triple, let $Z(A)$ be the centralizer of A, let $\partial_e A_1^*$ be the set of pure functionals of A and let* $\operatorname{MinPrimal} A$ *be the space of minimal primal ideals of A, equipped with the lower topology. For each element x in $\partial_e A_1^*$, let G_x be the unique element of* $\operatorname{MinPrimal} A$ *contained in the kernel of x. For each element T in $Z(A)$, let the complex valued function f_T on* $\operatorname{MinPrimal} A$ *be defined for each element x in $\partial_e A_1^*$ by*

$$T^* x = f_T(G_x) x$$

*Then the mapping $T \mapsto f_T$ is an isometric *-isomorphism from $Z(A)$ onto the space* $C^b(\operatorname{MinPrimal} A)$ *of continuous complex-valued bounded functions on* $\operatorname{MinPrimal} A$*, such that, for all a in A and G in* $\operatorname{MinPrimal} A$*,*

$$Ta + G = f_T(G)(a + G).$$

When A is a C*-algebra, the results of this section reduce to those found in [7], [73], [70], [61].

5.4 The Fell representation

Let A be a JB*-triple with primitive spectrum $\operatorname{Prim} A$ and let $\mathfrak{L}$ be the Lawson topology of A. In Section 5.1, the base space $\operatorname{Fell} A$ was introduced as the $\mathfrak{L}$-compact Hausdorff space $\overline{\operatorname{Prim} A}^{\mathfrak{L}}$, and it was shown that A has a natural representation onto a full triple of cross sections over $\operatorname{Fell} A$. The elements of $\operatorname{Fell} A$ are said to be the Fell ideals of A. By construction, $\operatorname{Fell} A$ possesses a $\mathfrak{L}$-dense subset of primitive ideals, and in the search for examples of densely standard JB*-triples, it is natural to ask when the range of the Fell representation is maximal. In this section, the theory developed in the rest of the chapter is applied to show that this happens exactly when $\operatorname{Prim} A$ is Hausdorff.

It will be convenient to exclude the element A from $\operatorname{Fell} A$, in order to work with proper primal ideals.

Lemma 5.19 *Let A be a JB*-triple and let $\operatorname{Fell}' A$ be the space of proper Fell ideals of A, equipped with the relative Lawson topology. Then $\operatorname{Fell}' A$ is a locally compact Hausdorff space and A is isometrically isomorphic to a full triple of cross sections over the base space $\operatorname{Fell}' A$.*

Proof. Let $\operatorname{Fell} A$ be the compact Hausdorff space of Fell ideals in the Lawson topology. Since $\operatorname{Fell}' A$ is an open dense subset of $\operatorname{Fell} A$, when equipped with the relative Lawson topology, $\operatorname{Fell}' A$ is a locally compact Hausdorff space. For all elements a in A, $\|a + A\|$ is zero, and, for $\varepsilon > 0$, the set

$$\{J \in \operatorname{Fell} A : \|a + J\| \in [\varepsilon, \|a\|]\}$$

is a compact subset of $\operatorname{Fell} A$ and hence of $\operatorname{Fell}' A$. Since $\cap \operatorname{Fell}' A$ is the zero ideal, the result follows from Lemma 4.32 and Lemma 5.5. ■

The representation described in Lemma 5.19 is called the *Fell representation* of A.

Lemma 5.20 *Let A be a JB*-triple. Then every Fell ideal is proper if and only if $\operatorname{Prim} A$ is Jacobson compact.*

Proof. Let $\mathfrak{L}$ be the Lawson topology on the lattice of norm-closed ideals of A. By Proposition 3.18, it is sufficient to prove that A lies in $\overline{\operatorname{Primal}' A}^{\mathfrak{L}}$, the $\mathfrak{L}$-closure of the set of proper primal ideals of A, if and only if A is a Fell ideal. Suppose that A lies in $\overline{\operatorname{Primal}' A}^{\mathfrak{L}}$ and let (I_μ) be a net in $\operatorname{Primal}' A$ $\mathfrak{L}$-convergent to A. Then a net

(P_μ) may be chosen in $\operatorname{Prim} A$ such that for all μ, P_μ dominates I_μ. For all elements a in A,

$$\|a + P_\mu\| \leq \|a + I_\mu\|.$$

It now follows from Proposition 4.32 that the net (P_μ) $\mathfrak{L}$-converges to A, and, hence, A is a Fell ideal. The converse implication is obvious. ∎

Theorem 5.21 *Let A be a JB*-triple and let* $\operatorname{Prim} A$ *be the primitive spectrum equipped with the Jacobson topology. Then the following are equivalent:*

(i) *the Fell representation $\hat{A}$ of A is maximal;*

(ii) $\operatorname{Prim} A$ *is Hausdorff.*

Proof. (i) $\Rightarrow$ (ii) Let $\operatorname{Fell}' A$ be the space of proper Fell ideals of A, equipped with the Lawson topology. Suppose that $\hat{A}$, the range of the Fell representation of A, is maximal. Then A is quasi-standard and, by Theorem 5.17, the space $\operatorname{Fell}' A$ in the Lawson topology coincides with $\operatorname{MinPrimal} A$, the set of minimal primal ideals of A, in the lower topology. It follows that the lower topology is Hausdorff on $\operatorname{Prim} A$.

(ii) $\Rightarrow$ (i) Suppose that $\operatorname{Prim} A$ is Hausdorff. Lemma 3.21 shows that $\operatorname{Fell}' A$ and $\operatorname{Prim} A$ coincide as topological spaces. Define the discrete equivalence relation $\sim$ on $\operatorname{Prim} A$ for P and Q in $\operatorname{Prim} A$ by $P \sim Q$ if and only if P and Q are equal. Then Theorem 5.14 implies that the Fell transform is maximal. ∎

It is also possible to give a direct proof of Theorem 5.21, adapting the argument used by Fell [42] in the C*-algebra case.

Important examples of JB*-triples with Hausdorff primitive spectrum include JB*-triples of constant finite rank ([20], Lemma 4.4) and abelian JB*-triples.

Chapter 6

Representations of JBW*-triples

In this chapter a Gelfand representation for the important class of JBW*-triples is investigated. In preparation for this investigation, in Section 6.1 the concept of Glimm ideals for a general JB*-triple is introduced. In Section 6.2 it is shown that JBW*-triples are quasi-standard, and that the base space of minimal primal ideals coincides with the space of Glimm ideals. In Section 6.3 it is shown that Type I JBW*-triples are densely standard.

6.1 Glimm ideals in JB*-triples

In this section the set of Glimm ideals in an arbitrary JB*-triple is introduced and some of its basic properties deduced. The relevance of Glimm ideals to the investigation into Gelfand representations comes from the fact that, for a JB*-triple possessing a complete tripotent, the set of Glimm ideals may be identified with the complete regularisation of the primitive spectrum. It is established that questions about Glimm and primitive ideals in a JB*-triple possessing a complete tripotent can be reduced to equivalent questions in a unital JB*-algebra.

Let A be a JB*-triple with primitive spectrum $\operatorname{Prim} A$. Let $Z(A)$ be the centralizer of A and let $\operatorname{Prim} Z(A)$ be the primitive spectrum of $Z(A)$. From Lemma 4.35, recall that for each norm-closed ideal J of A, the set

$$J^{\triangledown} = \{T \in Z(A) : TA \subseteq J\}$$

is a norm-closed ideal in $Z(A)$ and for each norm-closed ideal I in $Z(A)$, the set

$$I_{\triangle} = IA = \{Ta : T \in I, a \in A\}.$$

is a norm-closed ideal in A. The set $\operatorname{Glimm} A$ of *Glimm ideals* of A is defined by

$$\operatorname{Glimm} A = \{I_{\triangle} : I \in \operatorname{Prim} Z(A)\}.$$

Recall that, by Lemma 4.40, the complete regularisation $(\gamma, \gamma \operatorname{Prim} A)$ of $\operatorname{Prim} A$ with the Jacobson topology can be identified with the set of ideals:

$$\Omega(A) = \{\wedge\gamma(P) : P \in \operatorname{Prim} A\}.$$

Lemma 6.1 *Let A be a JB*-triple and let* $\operatorname{Glimm} A$ *be the set of Glimm ideals of A. Using the notation of Lemma 4.40, the set $\Omega(A)$ is a subset of* $\operatorname{Glimm} A$*. If $\Omega(A)$ is compact in the completely regular topology then $\Omega(A)$ and* $\operatorname{Glimm} A$ *coincide as sets and the map $G \mapsto G^{\triangledown}$ is a homeomorphism from* $\operatorname{Glimm} A$ *in the completely regular topology onto* $\operatorname{Prim} Z(A)$*, the primitive spectrum of the centralizer $Z(A)$ of A, with inverse $I \mapsto I_{\triangle}$.*

Proof. The result follows immediately from Theorem 4.41. ■

When A is a unital C*-algebra, $\operatorname{Glimm} A$ is the set of ideals considered by Glimm in [49], Section 4. To obtain an equivalent theory in the JB*-triple case, it is necessary to consider only JB*-triples which possess a complete tripotent. This is a large class including unital C*-algebras and JB*-algebras, and all JBW*-triples. Proposition 6.2 and its consequences allow the study of the primitive ideal space in such JB*-triples to be reduced to the unital JB*-triple case.

Proposition 6.2 *Let A be a JB*-triple possessing a complete tripotent u and let $A_2(u)$ be the Peirce-2 space of u. Let* $\operatorname{Prim} A$ *and* $\operatorname{Prim} A_2(u)$ *be the primitive spectra of A and the JB*-algebra $A_2(u)$ respectively, equipped with their Jacobson topologies. Then* $\operatorname{Prim} A$ *and* $\operatorname{Prim} A_2(u)$ *are compact and the map $\upsilon : J \mapsto J \cap A_2(u)$ a homeomorphism from* $\operatorname{Prim} A$ *onto* $\operatorname{Prim} A_2(u)$.

Proof. Assume that A possesses a pure functional x such that $A_2(u)$ is contained in $k_n(\ker_* x)$, the norm central kernel of x in A. Let $A_2^{**}(u)$ be the Peirce-2 space of u in A^{**} and let $k(\ker^* x)$ be the central kernel of x in A^{**}. Then, by Proposition 2.22,

$$A_2^{**}(u) = A_2(u)^{**} \subseteq k(\ker^* x).$$

Since u is complete in A, it is complete in A^{**}. Let $A_0^{**}(u)$ be the Peirce-0 space of u in A^{**}, let $k(A_0^{**}(u))$ be the central kernel of $A_0^{**}(u)$ and let $c(A_2^{**}(u))$ be the central hull of $A_2^{**}(u)$. Then, using Theorem 4.2,

$$\{0\} = A_0^{**}(u) = k(A_0^{**}(u)) = k(A_2^{**}(u)^{\perp}) = c(A_2^{**}(u))^{\perp}.$$

Therefore, as in Corollary 4.17,

$$A^{**} = c(A_2^{**}(u)) \subseteq k(\ker^* x) \subseteq \ker^* x \subseteq A^{**}.$$

which implies that x is equal to zero, yielding a contradiction. Thus $\operatorname{hull} A_2(u)$, the set of primitive ideals of A containing $A_2(u)$, is empty.

Applying [20], Proposition 3.3, the map $\upsilon : J \mapsto J \cap A_2(u)$ is a homeomorphism from $\operatorname{Prim} A$ onto $\operatorname{Prim} A_2(u)$. Since $A_2(u)$ is a unital JB*-algebra, $\operatorname{Prim} A_2(u)$ is compact. It follows that $\operatorname{Prim} A$ is also compact. ■

Corollary 6.3 *Let A be a JB*-triple possessing a complete tripotent, and adopt the notation of Lemma 6.1. Then* $\operatorname{Prim} A$ *is compact,* $\operatorname{Glimm} A$ *and* $\Omega(A)$ *coincide as sets and the completely regular and quotient topologies coincide on this set and are compact and Hausdorff.*

Proof. Let u be a complete tripotent in A and let $A_2(u)$ be the Peirce-2 space of A. By Proposition 6.2, $\mathrm{Prim}\, A$ is homeomorphic to $\mathrm{Prim}\, A_2(u)$, the primitive spectrum of the unital JB*-algebra $A_2(u)$. By the argument used in the case of C*-algebras [64], the primitive spectrum of a JB*-algebra is compact. Hence $\mathrm{Prim}\, A$ is compact. It follows that the quotient topology on $\Omega(A)$ is compact. The identity map from $\Omega(A)$ with the quotient topology to $\Omega(A)$ with the Hausdorff completely regular topology is continuous, and, hence, a homeomorphism. The result now follows from Lemma 6.1. ■

Corollary 6.4 is an analogue of a result for w*-closed ideals in JBW*-triples ([54], Theorem 4.2).

Corollary 6.4 *Let A be a JB*-triple possessing a complete tripotent u and let $A_2(u)$ be the Peirce-2 space of u. Then the map $\tilde{\upsilon} : J \mapsto J \cap A_2(u)$ is an order isomorphism from $\mathcal{ZI}_{\mathrm{n}}(A)$, the complete lattice of norm-closed ideals of A, onto $\mathcal{ZI}_{\mathrm{n}}(A_2(u))$, the complete lattice of norm-closed ideals of $A_2(u)$.*

Proof. For each element J in $\mathcal{ZI}_{\mathrm{n}}(A)$, let $\mathrm{hull}\, J$ be the set of primitive ideals of A containing J and let $\mathrm{hull}_{A_2(u)}(J \cap A_2(u))$ be the set of primitive ideals of $A_2(u)$ containing $J \cap A_2(u)$. By Proposition 6.2, the map $\upsilon : J \mapsto J \cap A_2(u)$ is a homeomorphism from $\mathrm{Prim}\, A$ onto $\mathrm{Prim}\, A_2(u)$. Let $\tilde{\upsilon}$ be the order isomorphism from $\mathcal{ZI}_{\mathrm{n}}(A)$ to $\mathcal{ZI}_{\mathrm{n}}(A_2(u))$ defined, for J in $\mathcal{ZI}_{\mathrm{n}}(A)$, by

$$\tilde{\upsilon}(J) = \bigcap \upsilon(\mathrm{hull}\, J).$$

Using the observation [20], that

$$\mathrm{hull}_{A_2(u)}(J \cap A_2(u)) = \{P \cap A_2(u) : P \in \mathrm{hull}\, J\}$$

it is clear that

$$\tilde{\upsilon}(J) = J \cap A_2(u).$$

■

Corollary 6.5 can be compared with Theorem 4.5.

Corollary 6.5 *Let A be a JB*-triple possessing a complete tripotent u and let $A_2(u)$ be the Peirce-2 space of u. Let $Z(A)$ and $Z(A_2(u))$ be the centralizers of A and $A_2(u)$ respectively. Then the map $T \mapsto T|_{A_2(u)}$ is a *-isomorphism from $Z(A)$ onto $Z(A_2(u))$.*

Proof. Let $\partial_e A_2(u)_1^*$ be the set of pure functionals of $A_2(u)$ and let y be an element of $\partial_e A_2(u)_1^*$. Then there exists a pure functional x of A extending y. For all a in $A_2(u)$ and T in $Z(A)$,

$$x(Ta) = \check{T}(y)x(a).$$

Thus,

$$T|_{A_2(u)}\check{}(y) = \check{T}(x).$$

Let $k_n^{A_2(u)}(\ker y)$ be the largest norm-closed ideal of $A_2(u)$ contained in $\ker y$ and let $k_n^A(\ker x)$ be the largest norm-closed ideal of A contained in $\ker x$. Let $\operatorname{Prim} A$ and $\operatorname{Prim} A_2(u)$ be the primitive spectra of A and $A_2(u)$ respectively, equipped with their Jacobson topologies. By Theorem 3.26 there exists f_T in $C^b(\operatorname{Prim} A)$, the space of continuous bounded complex-valued functions on $\operatorname{Prim} A$, and $f_{T|_{A_2(u)}}$ in $C^b(\operatorname{Prim} A_2(u))$, the space of continuous bounded complex-valued functions on $\operatorname{Prim} A_2(u)$, such that, for all such x and y,

$$f_T(k_n(\ker x)) = \check{T}(x) = \check{T}(y) = f_{T|_{A_2(u)}}(k_n^{A_2(u)}(\ker y)).$$

By [20],

$$k_n(\ker x) \cap A_2(u) = k_n^{A_2(u)}(\ker y),$$

and it follows that

$$f_T = f_{T|_{A_2(u)}} \circ \upsilon.$$

For g in $C^b(\operatorname{Prim} A_2(u))$, let S_g be the unique element of $Z(A_2(u))$ such that for all y in $\partial_e A_2(u)_1^*$,

$$g(k_n^{A_2(u)}(\ker y)) = \check{S}_g(y).$$

Then there is a *-isomorphism $\mu : Z(A) \mapsto Z(A_2(u))$ defined by

$$\mu(T) = S_{f_T \circ \upsilon^{-1}}$$

and

$$\mu(T) = T|_{A_2(u)}.$$

■

In Proposition 6.2, the space of primitive ideals of a JB*-triple A possessing a complete tripotent u was identified with the primitive spectrum of the unital JB*-algebra $A_2(u)$. Proposition 6.6 is the corresponding result for Glimm ideals.

Proposition 6.6 *Let A be a JB*-triple possessing a complete tripotent u and let $A_2(u)$ be the Peirce-2 space of u. Let $\Omega(Z(A))$ be the character space of the centralizer of A and let* $\operatorname{Glimm} A$ *and* $\operatorname{Glimm} A_2(u)$ *be the sets of Glimm ideals of A and $A_2(u)$ respectively, equipped with the topology of Corollary 6.3. Then the map $\tilde{\upsilon} : G \mapsto G \cap A_2(u)$ is a homeomorphism from* $\operatorname{Glimm} A$ *onto* $\operatorname{Glimm} A_2(u)$ *and for ω in $\Omega(Z(A))$, with kernel I_ω,*

$$\tilde{\upsilon}(I_\omega A) = I_\omega A_2(u).$$

Proof. Let $\operatorname{Prim} A$ be the primitive spectrum of A, let $(\gamma_A, \gamma \operatorname{Prim} A)$ be the complete regularisation of $\operatorname{Prim} A$ and let $\Omega(A)$ be the identification of $\gamma \operatorname{Prim} A$ with a set of ideals of A as in Lemma 4.40. Let $\operatorname{Prim} A_2(u)$, $(\gamma_A, \gamma \operatorname{Prim} A_2(u))$ and $\Omega(A_2(u))$ be the corresponding objects for the JB*-triple $A_2(u)$. By Proposition 6.2, the map $\upsilon : J \mapsto J \cap A_2(u)$ is a homeomorphism from $\operatorname{Prim} A$ onto $\operatorname{Prim} A_2(u)$. Observe that, for P and Q in $\operatorname{Prim} A$, $\gamma_A(P)$ equals $\gamma_A(Q)$ if and only if $\gamma_{A_2(u)}(\upsilon(P))$ equals $\gamma_{A_2(u)}(\upsilon(Q))$. Then a standard property of quotient spaces establishes the existence of a homeomorphism $\check{\upsilon}$ from $\Omega(A)$ onto $\Omega(A_2(u))$ defined, for P in $\operatorname{Prim} A$, by

$$\begin{aligned}\check{\upsilon}(\wedge\gamma_A(P)) &= \wedge\gamma_{A_2(u)}(\upsilon(P)) \\ &= \cap\{\upsilon(Q) : \gamma_A(Q) = \gamma_A(P)\} \\ &= (\wedge\gamma_A(P)) \cap A_2(u).\end{aligned}$$

By Corollary 6.3, the set $\Omega(A)$ coincides with $\operatorname{Glimm} A$ and $\Omega(A_2(u))$ coincides with $\operatorname{Glimm} A_2(u)$. Let ω be a character of $Z(A)$ and let a and T be elements of A and I_ω respectively such that Ta lies in $A_2(u)$. Then

$$Ta = P_2(u)Ta = TP_2(u)a.$$

Thus

$$(I_\omega A) \cap A_2(u) \subseteq I_\omega A_2(u).$$

The reverse inequality is obvious and completes the proof. ∎

6.2 Glimm ideals in JBW*-triples

In this section it will be shown that every JBW*-triple A is quasi-standard and that the base space $\operatorname{MinPrimal} A$ of minimal primal ideals coincides with $\operatorname{Glimm} A$, the set of Glimm ideals (Theorem 6.13). The following lemmas allow Theorem 6.13 to be deduced as an application of the main results of Chapter 5.

Lemma 6.7 *Let A be a JBW*-algebra with centre $Z(A)$. For ω in $\Delta(Z(A))$, the character space of $Z(A)$, let I_ω be the kernel of ω and let K_ω be the norm-closed ideal $I_\omega A$ of A. Then for every positive element a of A, there exists an element z_a in $Z(A)$, with Gelfand transform $\hat{z}_a$, such that, for all elements ω in $\Delta(Z(A))$,*

$$\|a + K_\omega\| = \hat{z}_a(\omega).$$

Proof. For each element z in $Z(A)$, let $\hat{z}$ be the Gelfand transform of z. For each element z in $Z(A)$ and ω in $\Delta(Z(A))$, observe that

$$z - \hat{z}(\omega)1 \in I_\omega \subseteq K_\omega.$$

and, in $Z(A)/K_\omega$,

$$z + K_\omega = \hat{z}(\omega)(1 + K_\omega). \qquad (6.2.1)$$

Suppose that b is an element of A such that, for all ω in $\Delta(Z(A))$, $b + K_\omega$ lies in the positive cone of A/K_ω. Then, for each pure state x of A, with restriction $\check{x}$ to $Z(A)$, there exists an element c in A such that $c + K_{\check{x}}$ is self-adjoint in $A/K_{\check{x}}$ and

$$b + K_{\check{x}} = (c + K_{\check{x}})^2.$$

Without loss of generality, c may be assumed to be self-adjoint in A. It follows that there exists an element d in $K_{\check{x}}$ such that

$$b = c^2 + d.$$

Therefore,

$$x(b) = x(c^2) + x(d) = x(c^2) \geq 0$$

and $\check{x}(b)$ is positive for all pure states x. By the Krein-Milman Theorem and [51], 1.2.5, b lies in A^+, the cone of positive elements of A.

Now let Z_a be the set

$$Z_a = \{z \in Z(A) : z \geq a\}.$$

It follows from the w*-closure of $Z(A)$ and A^+ that Z_a is a w*-closed subset of $Z(A)$. Let $z_1, \ldots, z_n$ be elements of Z_a with infimum z_0. Then, for ω in $\Delta(Z(A))$, using equation 6.2.1,

$$z_0 + K_\omega = \hat{z_0}(\omega)(1 + K_\omega) = ((\wedge_{j=1}^n \hat{z}_j(\omega)))(1 + K_\omega).$$

For a fixed element ω in $\Delta(Z(A))$, there exists an integer k in the range $1, \ldots, n$ such that, by equation 6.2.1,

$$((\wedge_{j=1}^{n}\hat{z}(\omega)))(1+K_\omega) = \hat{z}_k(\omega)(1+K_\omega) = z_k + K_\omega \geq a + K_\omega$$

By a previous remark, it follows that $z_0 - a$ is positive and z_0 is contained in Z_a. Let Λ be the directed set of finite subsets of Z_a, ordered by set inclusion. For each F in Λ, let

$$w_F = \bigwedge_{z\in F} z.$$

Then, from the above, $(w_F)_{F\in\Lambda}$ is a monotone decreasing net in Z_a, bounded below by a and it follows from [66], Lemma 1.7.4 that $(w_F)_{F\in\Lambda}$ converges in the w*-topology to its infimum z_a in $Z(A)$. Since Z_a is w*-closed, it follows that z_a is an element of Z_a and

$$z_a \geq a.$$

For all ω in $\Delta(Z(A))$,

$$0 \leq a + K_\omega \leq z_a + K_\omega = \hat{z_a}(\omega)(1+K_\omega)$$

which implies that

$$\|a + K_\omega\| \leq \hat{z_a}(\omega)\|1 + K_\omega\| = \hat{z_a}(\omega). \tag{6.2.2}$$

Assume that there exists an element ω_0 in $\Delta(Z(A))$ for which

$$\|a + K_{\omega_0}\| < \hat{z_a}(\omega_0).$$

Let

$$\varepsilon = (\hat{z_a}(\omega_0) - \|a + K_{\omega_0}\|)/3 > 0.$$

and let

$$U = \{\omega \in \Delta(Z(A)) : |\hat{z_a}(\omega) - \hat{z_a}(\omega_0)| < \varepsilon, \|a + K_\omega\| < \varepsilon + \|a + K_{\omega_0}\|\}.$$

Since A is unital, it follows from Lemma 6.1 and Corollary 6.3 that the map $\omega \mapsto K_\omega$ is a homeomorphism from $\Delta(Z(A))$ onto the set of ideals $\Omega(A)$ in the completely regular topology. By Corollary 5.15, $\omega \mapsto \|a + K_\omega\|$ is upper semi-continuous on $\Delta(Z(A))$. Thus U is an open neighbourhood of ω_0. By Urysohn's lemma, there exists an element z in $Z(A)_1^+$ such that $\hat{z}(\omega_0)$ is equal to 1 and $\hat{z}(\omega)$ is equal to 0 for all ω in $\Delta(Z(A))\backslash U$. For ω in U,

$$\begin{aligned} \|a + K_\omega\| &< \varepsilon + \|a + K_{\omega_0}\| \\ &= \varepsilon - 3\varepsilon + \hat{z}_a(\omega_0) \\ &\leq \hat{z}_a(\omega) - \varepsilon \\ &\leq \hat{z}_a(\omega) - \varepsilon\hat{z}(\omega). \end{aligned}$$

For ω in $\Delta(Z(A))\backslash U$, using equation 6.2.2,

$$\|a + K_\omega\| \leq \hat{z}_a(\omega) = \hat{z}_a(\omega) - \varepsilon\hat{z}(\omega).$$

It follows that

$$\|a + K_\omega\| 1_{\Delta(Z(A))} \leq z_a - \varepsilon z$$

Therefore, for all elements ω in $\Delta(Z(A))$

$$a + K_\omega \leq \|a + K_\omega\|(1 + K_\omega) \leq (z_a - \varepsilon z) + K_\omega.$$

and $z_a - \varepsilon z$ lies in Z_a. Since $\hat{z}$ is non-zero at ω_0, $z_a - \varepsilon z$ is strictly less than z_a, contradicting the fact that z_a is the infimum of Z_a. Therefore, by contradiction and equation 6.2.2, for all ω in $\Omega(Z(A))$,

$$\|a + K_\omega\| \geq \hat{z}_a(\omega) \geq \|a + K_\omega\|$$

which completes the proof. ∎

Lemma 6.8 *Let A be a JBW*-triple with centralizer $Z(A)$. For each element ω in*

$\Omega(Z(A))$, the character space of $Z(A)$, let I_ω be the kernel of ω. For every element a in A, there exists an element T_a in $Z(A)$ with Gelfand transform $\hat{T}_a$ such that, for all elements ω in $\Omega(Z(A))$

$$\|a + I_\omega A\| = \hat{T}_a(\omega).$$

The function $\omega \mapsto \|a + I_\omega A\|$ is continuous on $\Omega(Z(A))$.

Proof. By [31], Corollary 7.25, there exists a complete tripotent u in A such that u dominates the support tripotent $r(a)$ of a and a is a positive element of the JBW*-algebra $A_2(u)$. By Corollary 6.5 there exists an element T_a in $Z(A)$ such that $T_a u$ is the element z_a defined in Lemma 6.7. For all ω in $\Omega(Z(A))$, the natural embedding of $A_2(u)/(I_\omega A_2(u))$ into $A/(I_\omega A)$ is an injective triple homomorphism and by [10], Lemma 1,

$$\|a + I_\omega A\| = \|a + I_\omega A_2(u)\| = \hat{T}_a(\omega).$$

The continuity of $\omega \mapsto \|a + K_\omega\|$ on $\Omega(Z(A))$ is now immediate from the continuity of $\hat{T}_a$. ■

Lemma 6.9 *Let A be a JBW*-triple and let B be a w*-dense JB*-subtriple of A. Let $J_1, \ldots, J_n$ be norm-closed ideals of B. Then*

$$\bigcap_{j=1}^{n} \overline{J_j}^{w*} = \overline{\left(\bigcap_{j=1}^{n} J_j\right)}^{w*}.$$

Proof. By the separate w*-continuity of the triple product and the w*-density of B in A, for j equal to $1, 2, \ldots, n$, $\overline{J_j}^{w*}$ is a norm-closed ideal of A. Using Proposition 4.6 and the separate w*-continuity of the triple product,

$$\begin{aligned} \overline{J_1}^{w*} \cap \overline{J_2}^{w*} &= \left\{ \overline{J_1}^{w*}\, \overline{J_2}^{w*}\, \overline{J_1}^{w*} \right\} \\ &= \overline{\{J_1\, J_2\, J_1\}}^{w*} \\ &= \overline{J_1 \cap J_2}^{w*}. \end{aligned}$$

The result now follows by induction. ■

Lemma 6.10 *Let A be a JBW*-triple and let B be a w*-dense JB*-subtriple of A. For every element ω in $\Omega(Z(A))$, the character space of the centralizer $Z(A)$ of A, let I_ω be the kernel of ω. Then $B \cap I_\omega A$ is a primal ideal of B.*

Proof. Let ω be a character of $Z(A)$ and let $J_1, \ldots, J_n$ be norm-closed ideals of B with trivial intersection. Using Lemma 6.9, the w*-closed ideals $\overline{J_1}^{w*}, \ldots, \overline{J_n}^{w*}$ also have trivial intersection. Using the order isomorphism between the w*-closed ideals of A and the projections of $Z(A)$ [13], there exists projections $p_1, \ldots, p_n$ of $Z(A)$ such that for j equal to $1, \ldots, n$, $\overline{J_j}^{w*}$ coincides with $p_j A$ and $p_1 \ldots p_n$ is zero. Thus there exists an integer j in $1, \ldots, n$ such that $p_j(\omega)$ is zero. Then p_j lies in I_ω and $p_j A$ is a subspace of $I_\omega A$. Hence J_j lies in $I_\omega A \cap B$ and the proof is complete. ■

Corollary 6.11 *Let A be a JBW*-triple. Then every Glimm ideal of A is primal.*

Corollary 6.12 *Let A be a JB*-triple. Then for every Glimm ideal G in A^{**}, the bi-dual of A, $A \cap G$ is primal in A.*

It it now possible to prove the main theorem of this section.

Theorem 6.13 *Let A be a JBW*-triple. Then A is quasi-standard, and the base space* $\mathrm{MinPrimal}\, A$ *of minimal primal ideals coincides with the set* $\mathrm{Glimm}\, A$ *of Glimm ideals.*

Proof. By Corollary 6.3, $\mathrm{Glimm}\, A$ may be identified with the complete regularisation of the primitive spectrum of A and the quotient and completely regular topologies coincide and are compact and Hausdorff. By Lemma 6.1 and Lemma 6.8, for each a in A, the function $\rho_a : G \mapsto \|a+G\|$ is continuous on $\mathrm{Glimm}\, A$. Applying Theorem 5.14, (iii)⇒(iv), A is isometrically isomorphic to a maximal full triple of cross-sections over $\mathrm{Glimm}\, A$. By Corollary 6.11, every element of $\mathrm{Glimm}\, A$ is primal and A is therefore quasi-standard. By Theorem 5.17, $\mathrm{Glimm}\, A$ coincides with $\mathrm{MinPrimal}\, A$. ■

The following corollary is recorded for future reference.

Proposition 6.14 *Let A be a JBW*-triple and let J be a norm-closed ideal of A. Then J is primal if and only if J contains a Glimm ideal.*

Proof. The result follows from Lemma 3.16 and Theorem 6.13. ■

A natural question to ask is whether any additional theory exists for those prime and primal ideals in a JBW*-triple which are w*-closed. Lemma 6.15 uses Lemma 6.9 to answer this question.

Lemma 6.15 *Let A be a JBW*-triple, let $\mathcal{ZI}(A)$ be the complete Boolean algebra of w*-closed ideals of A and let $\mathcal{ZI}_{\mathrm{n}}(A)$ be the complete lattice of norm-closed ideals of A. Let P be an element of $\mathcal{ZI}(A)$. Then the following are equivalent:*

(i) *P is a prime element of $\mathcal{ZI}(A)$;*

(ii) *P is a primal element of $\mathcal{ZI}(A)$;*

(iii) *P is a maximal element of $\mathcal{ZI}(A)$ or P equals A;*

(iv) *P is a prime element of $\mathcal{ZI}_{\mathrm{n}}(A)$;*

(v) *P is a primal element of $\mathcal{ZI}_{\mathrm{n}}(A)$.*

Proof. (i)⇔(ii)⇔(iii) This is equivalence of (i), (ii) and (iii) in Lemma 3.9.

(i)⇒(iv) Let J_1 and J_2 be elements of $\mathcal{ZI}_{\mathrm{n}}(A)$ such that $J_1 \cap J_2$ is a subset of P. Then, by Lemma 6.9, $\overline{J_1}^{w*} \cap \overline{J_2}^{w*}$ is a subset of P. Thus at least one of $\overline{J_1}^{w*}$ and $\overline{J_2}^{w*}$ is a subset of P and the result follows.

(iv)⇒(v) This is a standard property of complete lattice.

(v)⇒(i) Since P is primal in $\mathcal{ZI}_{\mathrm{n}}(A)$, it is primal in $\mathcal{ZI}(A)$. ■

Corollary 6.16 will be of use in Chapter 7.

Corollary 6.16 *Let A be a JB*-triple with bidual A^{**}, let $\mathcal{ZI}_{\mathrm{n}}(A)$ be the complete lattice of norm-closed ideals of A and let P be a w*-closed ideal of A^{**} satisfying one of the equivalent conditions of Lemma 6.15. Then $A \cap P$ is a prime element of $\mathcal{ZI}_{\mathrm{n}}(A)$.*

Proof. Let J_1 and J_2 be elements of $\mathcal{ZI}_{\mathrm{n}}(A)$ such that $J_1 \cap J_2$ is a subset of $A \cap P$. By Lemma 6.9,

$$\overline{J_1}^{w*} \cap \overline{J_2}^{w*} \subseteq \overline{A \cap P}^{w*} \subseteq P$$

and therefore at least one of $\overline{J_1}^{w*}$ and $\overline{J_2}^{w*}$ is a subset of P. The result follows. ■

6.3 Glimm ideals in Type I JBW*-triples

In this section it is shown that every Glimm ideal of a Type I JBW*-triple is primitive. Combined with the results of Section 6.2, this shows that every Type I JBW*-triple is densely standard.

The discussion starts with a useful technical lemma.

Let A be a JBW*-triple and let u be a tripotent in A. Define the *central support* $C(u)$ of u to be the smallest M-projection P on A for which

$$Pu = u.$$

Lemma 6.17 *Let A be a JBW*-triple with centralizer $Z(A)$. Let $\Omega(Z(A))$ be the character space of $Z(A)$, and for ω in $\Omega(Z(A))$, let I_ω be the kernel of ω. For a tripotent u of A, define the subset S_u of $\Omega(Z(A))$ by*

$$S_u = \{\omega \in \Omega(Z(A)) : \|u + I_\omega A\| \neq 0\}.$$

Then S_u is w-clopen in $\Omega(Z(A))$ and the Gelfand transform of $C(u)$, the central support of u, coincides with χ_{S_u}, the characteristic function of S_u.*

Proof. Since u is a tripotent, for ω in $\Omega(Z(A))$, $\|u + I_\omega A\|$ can only take the values 0 and 1. Therefore,

$$S_u = \{\omega \in \Omega(Z(A)) : \|u + I_\omega A\| = 1\}$$

and it follows from Lemma 6.8 that S_u is w*-clopen. This implies that χ_{S_u} is a projection in $C_0(\Omega(Z(A)))$, the space of continuous complex-valued functions vanishing at infinity on $\Omega(Z(A))$.

Let $\widehat{C(u)}$ be the Gelfand transform of $C(u)$ in $C_0(\Omega(A))$. Let ω be an element of $\Omega(Z(A))$ such that $C(u)$ lies in I_ω. Then u lies in $I_\omega A$ and it follows that ω cannot lie in S_u. Therefore, for ω in S_u, $\omega(C(u))$ is non-zero. Since ω is a character and $C(\omega)$ a projection, it follows that $\widehat{C(u)}(\omega)$ equals 1. Hence $\widehat{C(u)} \geq \chi_{S_u}$.

Now let P be a projection in $Z(A)$ such that the Gelfand transform $\hat{P}$ dominates χ_{S_u}. Let x be an element of $\partial_e A_1^*$, the set of pure functionals of A. By Proposition 3.22, a character $\check{x}$ is defined for T in $Z(A)$ by the equation

$$T^* x = \check{x}(T)x.$$

If $\check{x}$ lies in S_u then

$$x(Pu) = \check{x}(P)x(u) = \hat{P}(\check{x})x(u) \geq \chi_{S_u}(\check{x})x(u).$$

Otherwise, if $\check{x}$ lies in the complement of S_u then

$$u \in I_{\check{x}} A \subseteq \ker x$$

and

$$x(Pu) = \check{x}(P)x(u) = 0 = x(u).$$

Thus, using the Krein-Milman Theorem, $Pu = u$, and it follows that $\chi_{S_u} \geq \widehat{C(u)}$. This completes the proof. ■

Corollary 6.18 *Let A be a JBW*-triple with centralizer $Z(A)$. Let ω be an element of $\Omega(Z(A))$, the character space of $Z(A)$ and let I_ω be the kernel of ω. Then, for every tripotent u in $I_\omega A$, the central support $C(u)$ of u lies in I_ω.*

Proof. Let u be a tripotent in $I_\omega A$ and let S_u be defined as in Lemma 6.17. Then ω lies in the complement of S_u. By Lemma 6.17, $C(u)$ is the characteristic function of S_u, and it follows that $\omega(C(u))$ equals zero, completing the proof. ■

Let A be a JB*-triple with centralizer $Z(A)$. A linear map $\phi : A \mapsto Z(A)$ is said to be a $Z(A)$-*functional* if, for all T in $Z(A)$ and a in A,

$$\phi(Ta) = T\phi(a).$$

Lemma 6.19 *Let A be a Type I JBW*-triple and let e be an abelian tripotent of A with central support Id_A and Peirce-2 space $A_2(e)$. Then A possesses a $Z(A)$-functional τ_e such that, for all a in A,*

$$\tau_e(P_2(e)a) = \tau_e(a).$$

*The map $\tau_e|_{A_2(e)}$ is an isometric *-isomorphism from $A_2(e)$ onto $Z(A)$ with inverse $T \mapsto Te$.*

Proof. By Theorem 4.5 there is an isometric isomorphism $\delta_e : Z(A) \mapsto A_2(e)$ defined by

$$\delta_e(T) = Te.$$

Define a linear map $\tau_e : A \mapsto Z(A)$ by

$$\tau_e = \delta_e^{-1} \circ P_2(e).$$

For T in $Z(A)$ and a in A,

$$\tau_e(Ta) = \delta_e^{-1}(TP_2(e)a) = \delta_e^{-1}\delta_e(T\tau_e(a)) = T\tau_e(a).$$

The remaining statements are immediate by construction. ■

Proposition 6.20 *Let A be a Type I JBW*-triple with centralizer $Z(A)$. Let e be an abelian tripotent in A with central support Id_A and Peirce-2 projection $P_2(e)$ and let $\tau_e : A \mapsto Z(A)$ be the $Z(A)$-functional of Lemma 6.19. Let μ be a character of $Z(A)$ with kernel I_μ, let K_μ be the ideal $I_\mu A$ of A and let x be the functional $\mu \circ \tau_e$ of A. Then:*

(i) *x is a pure functional of A;*

(ii) *K_μ is a subset of* $\ker x$*;*

(iii) *for a in A,*

$$P_2(e)a + K_\mu = x(a)(e + K_\mu).$$

Proof. (i) Let A_1^* be the unit ball of the dual A^* of A and let $\{e\}'$ be the w*-closed face of A_1^* given by

$$\{e\}' = \{y \in A_1^* : y(e) = 1\}.$$

Then

$$x(e) = \mu(\mathrm{Id}_A) = 1$$

and, for a in A,

$$|x(a)| \leq \|P_2(e)a\| = \|a\|.$$

Thus x lies in $\{e\}'$. For y in $\{e\}'$, define a bounded linear functional $\check{y}$ of $Z(A)$ by

$$\check{y}(T) = y(Te).$$

Then,

$$\check{y}(\mathrm{Id}_A) = y(e) = 1$$

and it follows that $\check{y}$ is a state of $Z(A)$. By [44], Proposition 1, for a in A,

$$y(a) = y(P_2(e)a) = y(\tau_e(a)e) = \check{y}(\tau_e(a)).$$

Observe that

$$\check{x}(T) = \mu(\tau_e(Te)) = \mu(T\tau_e(e)) = \mu(T).$$

Let x_1 and x_2 be elements of $\{e\}'$ and let λ be an element of $(0, 1)$ such that

$$x = \lambda x_1 + (1 - \lambda)x_2.$$

Then, for T in $Z(A)$,

$$\mu(T) = \check{x}(T) = \lambda \check{x}_1(T) + (1 - \lambda)\check{x}_2(T).$$

Since μ is a pure state of $Z(A)$,

$$\mu = \check{x}_1 = \check{x}_2.$$

Thus, for j equal to 1 and 2,

$$x_j(a) = \check{x}_j(\tau_e(a)) = \mu(\tau_e(a)) = x(a)$$

and it follows that x is a pure functional of A.

(ii) Let T be an element of I_μ and let a be an element of A. Then,

$$x(Ta) = \mu(\tau_e(Ta)) = \mu(T)\mu(\tau_e(a)) = 0.$$

Therefore K_μ is a subset of $\ker x$.

(iii) Let p be a projection in the JBW*-algebra $A_2(e)$, the Peirce-2 space of e. Then there exists a projection P in $Z(A)$ such that

$$Pe = p.$$

Since μ is a character, either P or $I - P$ lies in I_μ. Thus, there exists λ_p in $\{0, 1\}$ such that

$$p + K_\mu = \lambda_p e + K_\mu.$$

By [51], Proposition 4.2.3, the JBW*-algebra $A_2(e)$ is the norm-closed linear span of its projections. Therefore, for all a in A, there exists λ_a in $\mathbb{C}$ such that

$$P_2(e)a + K_\mu = \lambda_a e + K_\mu.$$

By (ii), K_μ is a subset of $\ker x$. Therefore,

$$x(P_2(e)a) = \lambda_a x(e).$$

It follows that

$$\lambda_a = x(a).$$

■

Proposition 6.21 *Let A be a Type I JBW*-algebra with centralizer $Z(A)$, let e be an abelian projection in A with central support* Id_A *and Peirce-2 projection $P_2(e)$. Let μ be a character of $Z(A)$ with kernel I_μ, let x be the functional $\mu \circ \tau_e$, let* $\ker x$ *be the kernel of x in A and let $k_n(\ker x)$ be the norm central kernel of x in A. Let K be the set*

$$K = \{a \in A : P_2(e)Q(b)a \in K_\mu \, \forall b \in A\},$$

let J be the set

$$J = \{a \in A : x(Q(b)a) = 0 \, \forall b \in A\}$$

and let K_μ be the ideal

$$K_\mu = I_\mu A.$$

Then

$$K_\mu = K = J = k_n(\ker x).$$

Proof. Let a be an element of K_μ. Since K_μ is an ideal, it follows from Theorem 4.1 that for all b in A, $P_2(e)Q(b)a$ lies in K_μ. Hence a lies in K and K_μ is a subset of K.

Let a be an element of K. By Proposition 6.20, for all b in A, $P_2(e)Q(b)a$ lies in $\ker x$, and therefore

$$x(Q(b)a) = x(P_2(e)Q(b)a) = 0.$$

It follows that a lies in J and K is a subset of J.

By Theorem 4.15, J coincides with $k_n(\ker x)$.

Let a be an element of J. Then by Proposition 6.20, for all b in A,

$$P_2(e)Q(b)a + K_\mu = x(Q(b)a)(e + K_\mu) = 0,$$

and hence $P_2(e)Q(b)a$ lies in K_μ. Therefore a lies in K and J is a subset of K.

Let v be a projection in K. By the comparison theorem of projections [51], 5.2.13, there exists symmetries s and t in A and a projection P in $Z(A)$ such that

$$Q(s)(Pe) \leq Pv \qquad Q(t)(P^\perp v) \leq P^\perp e.$$

Then

$$Pe = P_2(e)Pe \leq P_2(e)Q(s)Pv$$

and since Pv lies in K, $P_2(e)Q(s)Pv$ and hence Pe lie in K_μ. Assume for a contradiction that P does not lie in I_μ. Then $I - P$ lies in I_μ and hence $e - Pe$ lies in K_μ. This implies that e lies in K_μ, and by Corollary 6.18, Id_A, the central support of e, lies in I_μ, providing the required contradiction. Therefore P lies in I_μ, from which it follows that Pv lies in K_μ. Since it is also the case that

$$P^\perp v \leq Q(t)P^\perp e \leq Q(t)e,$$

$P^\perp v$ is a projection in the Peirce-2 space $A_2(Q(t)e)$ of the projection $Q(t)e$ of A. Hence

$$\begin{aligned} P^\perp v &= Q(Q(t)e)P^\perp v \\ &= Q(t)Q(e)Q(t)P^\perp v \end{aligned}$$

Since $P^\perp v$ lies in K, $P_2(e)Q(t)P^\perp v$ and hence $P^\perp v$ lie in K_μ. Therefore,

$$v = Pv + P^\perp v$$

lies in K_μ. The norm-closed ideal K is generated by its projections, and hence K is a subset of K_μ. This completes the proof. ■

Proposition 6.22 is now immediate from Proposition 6.20 and Proposition 6.21.

Proposition 6.22 *Let A be a Type I JBW*-algebra. Then every Glimm ideal in A is primitive.*

Lemma 6.23 *Let A be a JB*-triple and let u be a complete tripotent with Peirce-2 space $A_2(u)$, such that every Glimm ideal of the JB*-algebra $A_2(u)$ is primitive. Then every Glimm ideal of A is primitive.*

Proof. Let G be a Glimm ideal of A. By Proposition 6.6, $G \cap A_2(u)$ is a Glimm ideal of $A_2(u)$, and therefore a primitive ideal of $A_2(u)$. By Proposition 6.2, A possesses a primitive ideal P such that $P \cap A_2(u)$ coincides with $G \cap A_2(u)$. By Corollary 6.4, G and P coincide. ■

The main result of this section is now immediate from Proposition 6.22, Lemma 6.23 and Theorem 6.13.

Theorem 6.24 *Let A be a Type I JBW*-triple. Then every Glimm ideal of A is a primitive ideal of A and A is densely standard.*

Chapter 7

Factorial Functionals and Primal Ideals

In this chapter, the study of factorial functionals on JB*-triples is initiated. After some technical results are established, the connections between factorial functionals and primal ideals are explained. A number of applications to JB*-triple structure theory are given, as described below.

In Section 7.1, the definition of a factorial functional on an arbitrary Banach space is given and this definition is interpreted in terms of the ideal and local order structure of a JB*-triple. In Section 7.2, a special class of factorial functionals, the Type I factorial functionals, is defined. This class is accessible because its elements can be constructed from σ-convex sums of quasi-equivalent pure functionals. In Section 7.3, an important technical result is established, characterising the primal ideals of a JB*-algebra in terms of w*-density properties of the factorial states in faces of the state space. In Theorem 7.13, the important part of this result is then extended to JB*-triples. Whilst Theorem 7.13 is a pleasing result in its own right, the real interest comes from the applications explored in the remainder of this chapter. The first is described in Section 7.4, where the relationship between factorial functionals and primal ideals is illustrated by the existence of a natural map from the set of factorial functionals into the primal ideals. In Section 7.5 some more properties of this mapping are explored. In Section 7.6 the use of Theorem 7.13 to decompose a key stage in the proof of the Glimm Stone-Weierstrass Theorem for JB*-triples is discussed. The results of this section include a characterisation of the prime JB*-triples and necessary conditions for a JB*-triple to be anti-liminal.

In Section 7.8 a characterisation of the pure functional space of a continuous JBW*-triple in terms of the characters of the centralizer is presented. To achieve this, a map from the dual space of a JB*-triple into the dual space of its centralizer is introduced and studied in Section 7.7. This map takes the place of the restriction of positive functionals to the centre in the unital C*-algebra case.

7.1 Definitions

Let A be a Banach space with dual space A^* having unit ball A_1^* with surface ∂A_1^*. An element x of ∂A_1^* is said to be *factorial* if the smallest L-summand L_x containing x, is a factor. The elements of $\partial_e A_1^*$, the set of pure functionals of A, are important examples of factorial functionals. The set of factorial functionals is denoted by $\partial_f A_1^*$. Two elements x and y in A^* are said to be *quasi-equivalent* if L_x and L_y coincide.

Factorial functionals have been studied studied for real Banach spaces in [28] and for GM-spaces in [27] under the name *globally primary functionals*. The main result of this section (Theorem 7.3) describes some equivalent conditions for functionals to be factorial. In particular, it will be shown that, for a JB-algebra, the factorial functionals are the *locally primary functionals* studied by Wils [78]. The results will be proved for normal functionals on a JBW*-triple, greater generality than is required for this purpose. The discussion begins with some information about the order structure of the predual of a JBW-algebra.

Let V be a partially ordered vector space, let $\operatorname{End} V$ be the algebra of all linear maps $T : V \mapsto V$ and define the *ideal centre* of V, denoted by $\mathcal{O}(V)$, to be the subalgebra

$$\mathcal{O}(V) = \{T \in \operatorname{End} V : -\lambda \mathrm{Id}_V \leq T \leq \lambda \mathrm{Id}_V \text{ for some } \lambda \in \mathbb{R}^+\}.$$

Proposition 7.1 *Let A be a JBW-algebra with predual A_*, let $\mathcal{O}(A_*)$ be the ideal centre of A_* and let $Z(A)$ be the centralizer of A. For T in $\mathcal{O}(A_*)$, let T^* be the dual operator on A. Then the map $T \mapsto T^*$ is an isometric isomorphism of $\mathcal{O}(A_*)$ onto $Z(A)$.*

Proof. The result is the combination of [2], 5.7, 6.11, 6.12. ■

Proposition 7.2 *Let A be a JBW-algebra, let x be a faithful normal state, let A_* be the predual of A, let A_*^+ be the cone of positive elements of A_* and let $\mathcal{O}(A_*)$ be the ideal centre of A_*. For an element x in A_*^+, let V_x be the subspace of A_* generated by the face of A_*^+ generated by x and let $\mathcal{O}(V_x)$ be the ideal centre of V_x. Then, the map $T \mapsto T|_{V_x}$ is an isometric order and algebra isomorphism from $\mathcal{O}(A_*)$ onto $\mathcal{O}(V_x)$.*

Proof. Let A^* be the dual of A, let A^{**} be the bidual of A and let A^*_+ be the cone of positive elements in A^*. By Theorem 4.8, there exists an M-projection P_A on A^{**} such that A_* may be canonically identified with the L-summand $(P_A)_*A^*$ of A^*. Under this identification, V_x is identified with the subspace of A^* generated by the face of A^*_+ generated by x. Define the projection P_x of $\mathcal{O}(A^*)$, the ideal centre of A^*, by

$$P_x = \bigwedge\{P \in \mathcal{O}(A^*) : Px = x\}.$$

By [29], Lemma 3, Corollary 4 and [69], Theorem 5.1, the map $T \mapsto T|_{V_x}$ is an isometric algebraic and order isomorphism from $P_x\mathcal{O}(A^*)$ onto $\mathcal{O}(V_x)$. Now, $P_A - P_x$ is a projection in $P_A\mathcal{O}(A^*)$, and, by Proposition 7.1, there exists a projection P in $Z(A)$, the centralizer of A, such that, for all y in A_*,

$$(P1)(y) = 1((P_A - P_x)y).$$

In particular,

$$(P1)^2(x) = 1((P_A - P_x)x) = 1(0) = 0.$$

Since x is faithful on A, $P1$ equals 0. It follows that P_A and P_x coincide and the proof is complete. ■

As Theorem 7.3 shows, there are many equivalent ways of considering the central decompositions associated with a normal functional on a JBW*-triple, and hence many equivalent conditions for a functional to be factorial.

Theorem 7.3 *Let A be a JBW*-triple and let x be an element of unit norm in $A_{*,1}$, the unit ball of the predual A_* of A. Let L_x be the smallest L-summand of A^* containing x, let $e(x)$ be the support tripotent of x, let $A_2(e(x))$ be the Peirce-2 space of $e(x)$, let $A_2(e(x))_*$ be its predual and let $c(A_2(e(x)))$ be its central hull. Let V_x^+ be the face of the positive cone of $A_2(e(x))_*$ generated by x, let V_x be the order unit space generated in $A_2(e(x))_*$ by V_x^+ and let $\mathcal{O}(V_x)$ be the ideal centre of V_x. For each projection p in $A_2(e(x))$, let $\{p\}_\prime$ be the norm-closed face*

$$\{p\}_\prime = \{y \in A_{*,1} : y(p) = 1\}$$

of $A_{*,1}$. Then L_x coincides with the smallest L-summand of A_* containing x and there are order-isomorphisms between the following complete Boolean algebras:

(i) *the set of L-projections of L_x;*

(ii) *the set of M-projections of $c(A_2(e(x)))$;*

(iii) *the set of M-projections of $A_2(e(x))$;*

(iv) *the set of central projections of $A_2(e(x))$;*

(v) *the set of split faces of $\{e(x)\}_\prime$;*

(vi) *the set of idempotents of $\mathcal{O}(V_x)$.*

Proof. By Theorem 4.8, A_* is an L-summand of A^* containing x and therefore L_x is the smallest L-summand of A_* containing x. It follows from Theorem 4.7 that the dual space of L_x may be identified with $c(A_2(e(x)))$. Hence, the map $P \mapsto P^*$ is an isomorphism from (i) to (ii) (Proposition 3.5). By Theorem 4.5, the map $T \mapsto T|_{A_2(e(x))}$ is a *-isomorphism from $Z(c(A_2(e(x))))$, the centralizer of $c(A_2(e(x)))$, onto $Z(A_2(e(x)))$, the centralizer of $A_2(e(x))$. Hence, the map $P \mapsto P|_{A_2(e(x))}$ is an isomorphism from (ii) to (iii). By Proposition 4.4, the map $P \mapsto Pe(x)$ is an isomorphism from (iii) to (iv). When $A_2(e(x))_*$ is identified with the subspace $\{y \in A_* : \|y|_{A_2(e(x))}\| = \|y\|\}$ of A_*, $\{e(x)\}\prime$ is identified with the normal state space of the JBW*-algebra $A_2(e(x))$. The isomorphism from (iv) to (v) was described in Proposition 2.10 as the map $p \mapsto \{p\}\prime$. Finally, let $A_2(e(x))_{*,sa}$ be the self-adjoint part of the predual $A_2(e(x))_*$ of $A_2(e(x))$. By Proposition 7.2, $\mathcal{O}(V_x)$ may be identified with $\mathcal{O}(A_2(e(x))_{*,sa})$, the ideal centre of $A_2(e(x))_{*,sa}$ and by Proposition 7.1, $\mathcal{O}(A_2(e(x))_{*,sa})$ may be identified with $Z(A_2(e(x)))_{sa}$, the self-adjoint part of the centralizer of $A_2(e(x))$. Thus (vi) is isomorphic to (iii). ■

It follows from the definition that a functional x of unit norm on a JB*-triple A is factorial if and only if the Boolean algebras of Theorem 7.3 are trivial. In particular, x is factorial if and only if it is (locally) primary in the sense of Wils [78] in the partially ordered real vector space $A_2^{**}(e(x))_{*,sa}$.

Example 7.4 *Let A be an abelian JB*-triple. Then*

$$\partial_e A_1^* = \partial_f A_1^*$$

Proof. Let x be an element of $\partial_f A_1^*$. Then x lies in $\partial_e A_1^*$ since

$$A_2^{**}(e(x)) = Z(A_2^{**}(e(x))) \cong \mathbb{C}.$$

■

7.2 Type I factorial states

The Type I factorial functionals are of particular importance in the theory because they include a class of functionals which is w*-dense in all the factorial functionals (Corollary 7.16) yet are easily constructed from pure functionals (Proposition 7.10).

A JBW*-triple is said to be of *Type I* if it possesses an abelian tripotent with central support equal to the identity operator. A JBW*-triple factor is of Type I if and only if it possesses a minimal tripotent.

Lemma 7.5 *Let A be a JBW*-algebra which is a factor. Then A is a Type I JBW*-triple if and only if A possesses a minimal projection.*

Proof. Suppose that A possesses a minimal tripotent u with Peirce-2 space $A_2(u)$. Then, by [30], Theorem 2.3, there exists a projection p of A such that

$$A_2(u) = \mathbb{C}u = \mathbb{C}p = pAp.$$

Therefore p is a minimal projection in A. The converse is obvious. ∎

A JBW*-triple is said to be *atomic* if it is the w*-closed linear span of its minimal tripotents.

Lemma 7.6 *Let the JBW*-triple A be a non-zero factor. Then A is of Type I if and only if A is atomic.*

Proof. Let A be a factor of Type I. Since A contains a minimal tripotent, by [44], Theorem 2, A contains a non-zero w*-closed atomic ideal J. Since A is a factor, A coincides with J and is atomic. Conversely, if A is an atomic factor, it possesses a minimal tripotent and is therefore a factor of Type I. ∎

Lemma 7.7 *Let A be a JBW*-triple and let J be a w*-closed inner ideal of A. Then c(J), the central hull of J, is a Type I factor if and only if J is a Type I factor.*

Proof. Suppose that $c(J)$ is of Type I, and let v be any non-zero tripotent of J. Since $c(J)$ is the w*-closed linear span of its minimal tripotents, there exists a minimal tripotent u in $c(J)$ such that $P_2(v)u$ is non-zero. Furthermore, $P_2(v)u$ lies in J, and, by [44], Proposition 6, is a scalar multiple of a minimal tripotent. The converse is obvious. ∎

Let A be a JBW*-algebra factor and let k be a cardinal number. Then A is said to be of Type I_k if the unit is the sum of k minimal projections. The next lemma follows from [51], Section 5.3 and [35], Theorem 3.2.

Lemma 7.8 *Let A be a JBW*-triple, let x be a normal functional on A of unit norm such that the Peirce-2 space $A_2(e(x))$ of the support tripotent $e(x)$ of x is a factor of Type I and let* $\mathrm{Card}\,\mathbb{N}$ *be the cardinality of* $\mathbb{N}$. *Then there exists a unique element k of* $\mathbb{N} \cup \{\mathrm{Card}\,\mathbb{N}\}$ *such that $A_2(e(x))$ is of Type I_k.*

In the situation described in Lemma 7.8, x is said to be a factorial functional of Type I_k.

In [12], Archbold and Batty defined a state x of the C*-algebra A to be of Type $\mathrm{I_n}$ if $\pi_x(A)'$ is a Type $\mathrm{I_n}$ W*-algebra, where (π_x, H_x, ξ_x) is the GNS construction for x and $\pi_x(A)'$ is the commutant of $\pi_x(A)$ in $B(H_x)$, the W*-algebra of bounded linear operators on H_x. To see the connection with the definition given here, recall that, for each element Γ in $\pi_x(A)'_{sa}$ there exists an element y_Γ in V_x defined by

$$y_\Gamma(a) =< \Gamma\pi_x(a)\xi_x, \xi_x >_x,$$

the map $\Gamma \mapsto y_\Gamma$ is an order isomorphism from $\pi_x(A)'_{sa}$ onto V_x and the map $\Gamma \mapsto e(y_\Gamma)$ is an injection from the minimal elements of the atomic lattice $\mathcal{P}(\pi_x(A)'_{sa})$ into the minimal elements of the atomic lattice $\mathcal{P}(A_2^{**}(e(x)))$.

The remainder of this section focuses on describing the Type I factorial functionals.

Let $(x_\lambda)_{\lambda\in\Lambda}$ be a family of elements in a Hausdorff topological vector space (X, τ). Let $\mathcal{F}$ be the set of finite subsets of Λ, directed by set inclusion. Then $(\sum_{\lambda\in F} x_\lambda)_{F\in\mathcal{F}}$ is a net in X. When this net has a limit, the family $(x_\lambda)_{\lambda\in\Lambda}$ is said to be summable and the necessarily unique limit is said to be the sum, denoted by $\sum^\tau_{\lambda\in\Lambda} x_\lambda$. When Λ equals $\mathbb{N}$, the series converges to the sum. We say that an element x of X is a σ-convex sum of elements $(x_j)_{j=1}^\infty$ in X if it has the form

$$x = \sum_{j=1}^{\infty} \lambda_j x_j$$

where (λ_j) is a sequence of positive real numbers with sum 1.

Let $(u_\lambda)_{\lambda\in\Lambda}$ be a family of pairwise orthogonal tripotents in a JBW*-triple. Then, by [11], Proposition 3.4 (iv), for each finite subset F of Λ

$$\bigvee_{\lambda\in F} u_\lambda = \sum_{\lambda\in F} u_\lambda$$

and, by [11], Proposition 3.8 (iii), the supremum and sum of the family exist and satisfy

$$\bigvee_{\lambda\in\Lambda} u_\lambda = \sum_{\lambda\in\Lambda} u_\lambda.$$

Let A be a JBW*-triple with predual A_*. Elements x and y in A_* with support tripotents $e(x)$ and $e(y)$ in A are said to be *L-orthogonal* if $e(x)$ and $e(y)$ are orthogonal.

Lemma 7.9 *Let A be a JBW*-triple with pre-dual A_* and let k be either the cardinality of $\mathbb{N}$ or an element of $\mathbb{N}$. Let $(x_j)_{j=1}^k$ be a sequence of elements in $\partial A_{*,1}$, the surface of the unit ball of A_*, let $(\lambda_j)_{j=1}^k$ be a sequence of real numbers in the interval $(0,1)$ summing to 1 and such that the series $\sum_{j=1}^k \lambda_j x_j$ converges to an element x of A_*. For each j in $1,\ldots,k$, let $e(x_j)$ be the support tripotent of x_j and let $e(x)$ be the support tripotent of x. Then $\bigvee_{j=1}^k e(x_j)$ exists if and only if x has unit norm. In this case*

$$e(x) = \bigvee_{j=1}^k e(x_j).$$

Furthermore, if the sequence (x_j) is pairwise L-orthogonal then

$$e(x) = \sum_{j=1}^k e(x_j).$$

Proof. For each tripotent u of A, let $\{u\}_\prime$ be the proper norm-closed face

$$\{u\}_\prime = \{y \in A_{*,1} : y(u) = 1\}$$

of $A_{*,1}$. Suppose that an upper bound u for the sequence $(e(x_j))$ exists. Define sequences (α_n) and (y_n) for $n \leq k$ by

$$\alpha_n = \sum_{j=1}^{n} \lambda_j \qquad y_n = \sum_{j=1}^{n} \lambda_j x_j.$$

Then, the sequence (y_n/α_n) norm-converges to x and lies in the convex hull of $x_1, \ldots, x_n$, which is a subset of $\{u\}_\prime$ by Theorem 2.20. Hence x lies in $\{u\}_\prime$ and has unit norm. Furthermore, again by Theorem 2.20, u dominates $e(x)$. Conversely, suppose that x is of unit norm. For any l between 1 and k, define,

$$z_l = \sum_{\substack{j=1 \\ j \neq l}}^{k} \frac{\lambda_j}{1 - \lambda_l} x_j.$$

Then, as a σ-convex sum of elements of $\partial A_{*,1}$, z_l lies in $A_{*,1}$ and

$$x = \lambda_l x_l + (1 - \lambda_l) z_l$$

Since x lies in the face $\{e(x)\}_\prime$ of $A_{*,1}$, it follows that x_l and z_l also lie in $\{e(x)\}_\prime$. Thus, by Theorem 2.20, $e(x_l)$ is dominated by $e(x)$. It follows that $e(x)$ is an upper bound for $e(x_1), \ldots, e(x_k)$. Combining this fact with the first part of the proof,

$$e(x) = \bigvee_{j=1}^{k} e(x_j).$$

If the sequence (x_j) is pairwise L-orthogonal, then by [11],

$$e(x) = \sum_{j=1}^{k} e(x_j).$$

■

The following result is the JBW*-algebra equivalent of [12], Proposition 2.1.

Proposition 7.10 *Let A be a JBW*-triple with predual A_* and let x be a normal functional of A of unit norm. Let $e(x)$ be the support tripotent of x, let $A_2(e(x))$ be* *the Peirce-2 space of* $e(x)$ *and let* $\partial_e\{e(x)\}_{\prime}$ *denote the set of extreme points of the norm-closed face*

$$\{e(x)\}_{\prime} = \{y \in A_{*,1} : y(e(x)) = 1\}$$

of the unit ball $A_{*,1}$ *of* A*. Then the following results hold.*

(i) *If the JBW*-algebra* $A_2(e(x))$ *is a factor of Type* I_k *for some* k *a natural number or the cardinality of* $\mathbb{N}$*, then there exists* k *elements* $(x_j)_{j=1}^k$ *of* $\partial_e\{e(x)\}_{\prime}$ *such that* x *is the convex or* σ*-convex sum of the sequence* $(x_j)_{j=1}^k$ *and the* $(x_j)_{j=1}^k$ *are quasi-equivalent to* x*.*

(ii) *Let* (x_j) *be a sequence of quasi-equivalent elements of* $\partial_e A_{*,1}$ *such that* x *is the* σ*-convex sum of* (x_j)*. Then* x *is quasi-equivalent to each element of* (x_j)*, the set* S *of elements of* (x_j) *with non-zero coefficient is a subset of* $\partial_e\{e(x)\}_{\prime}$ *and* $A_2(e(x))$ *is a factor of Type* I_n *for some cardinal* n *less than or equal to the cardinality of* S*.*

Proof. (i) Suppose that $A_2(e(x))$ is a Type I_k factor. Then, by Lemma 7.6, $A_2(e(x))$ is an atomic JBW*-algebra, and, by [3], Proposition 5.6, there exists a sequence (y_j) of pairwise orthogonal elements of $\partial_e\{e(x)\}_{\prime}$ and a sequence (λ_j) of elements of $[0,1]$ summing to 1, such that x is the σ-convex sum

$$x = \sum_{j=0}^{\infty} \lambda_j y_j.$$

For j in $\mathbb{N}$, let $e(y_j)$ be the support tripotent of y_j in A. By Lemma 7.9,

$$e(x) = \sum_{j\in\mathbb{N},\lambda_j\neq 0} e(y_j)$$

and since this is a sum of minimal projections, it follows from Lemma 7.8 that the sum has k terms. Let $(x_j)_{j=1}^k$ be the subsequence of (y_j) with non-zero coefficients. Since $c(A_2(e(x)))$, the central hull of $A_2(e(x))$, is a factor, for each j in $1,\ldots,k$, $c(A_2(e(x_j)))$, the central hull of $A_2(e(x_j))$, coincides with $c(A_2(e(x)))$ and x_j is quasi-equivalent to x. Thus $(x_j)_{j=1}^k$ is the desired sequence.

(ii) Let (x_j) be a sequence of quasi-equivalent elements of $\partial_e A_{*,1}$ and let (λ_j) be a sequence of elements of $[0,1]$ summing to 1, such that x is the σ-convex sum

$$x = \sum_{j=0}^{\infty} \lambda_j x_j.$$

Let L be the minimal L-summand of A_* containing the sequence (x_j). Then x is an element of L, and by Theorem 7.3, $A_2(e(x))$ is a factor. By Lemma 7.9, for each j in $\mathbb{N}$ such that λ_j is non-zero, x_j lies in $\partial_e\{e(x)\}_{\prime}$, and $e(x_j)$, the support tripotent of x_j is a projection in the JBW*-algebra $A_2(e(x))$. By Lemma 7.9,

$$e(x) = \bigvee\{e(y) : y \in S\}.$$

By Lemma 7.8, $A_2(e(x))$ is a Type I_n factor for n an element of $\mathbb{N}$ or the cardinality of $\mathbb{N}$, and by Lemma 7.6, $A_2(e(x))$ is atomic. Let k be the cardinality of the set S. If k is equal to the cardinaltiy of $\mathbb{N}$, then, necessarily, $n \leq k$. If k is an element of $\mathbb{N}$, then, [3], Lemma 5.1 implies that $n \leq k$. ■

7.3 Primal ideals in JBW*-triples

In this section a characterisation of primal ideals in JB*-algebras in terms of the factorial state space is given. Part of this characterisation is then extended to JB*-triples. In addition to being of interest in their own right, the results of this section have many important consequences which are explored in subsequent sections.

Theorem 7.11 *Let A be a JB*-algebra and let K be a norm-closed ideal of A. Let $S(A)$ be the state space of A, let $\partial_f S(A)$ be the set of factorial states and let $\partial_f^f S(A)$ be the set of factorial states of Type* $\mathrm{I_n}$ *with n finite. Let K° be the topological annihilator of K. Then the following are equivalent:*

(i) *K is primal;*

(ii) $S(A) \cap K^\circ \subseteq \overline{\partial_f^f S(A)}^{w*}$;

(iii) $S(A) \cap K^\circ \subseteq \overline{\partial_f S(A)}^{w*}$.

Proof. (i)⇒(ii): Let $\partial_e S(A)$ denote the pure state space of A and for any element y of A^*, let J_y denote the norm central kernel of y. For j equal to $1, \ldots, n$, let x_j be an element of $\partial_e S(A) \cap K^\circ$ and let x be the convex sum $\sum_{j=1}^n \lambda_j x_j$. Let U be a w*-open convex neighbourhood of the origin in A^* and for j equal to $1, \ldots, n$, let the subset V_j of $\mathrm{Prim}\, A$ be given by

$$V_j = \{J_y : y \in \partial_e S(A), y - x_j \in U\}.$$

By [50], Theorem 4.1, V_j is open in $\mathrm{Prim}\, A$. Let J_j be the norm-closed ideal of A such that

$$V_j = \{P \in \mathrm{Prim}\, A : J_j \not\subseteq P\}.$$

Then, for j equal to $1, \ldots, n$, x_j is non-zero on J_j and therefore J_j is not a subset of K. Since K is primal, this implies that the ideal

$$J = \bigcap_{j=1}^n J_j$$

is non-zero. Let w be a pure state of J, let $e(w)$ be its support projection in A^{**} and let $c(A_2^{**}(e(w)))$ be the central hull of the Peirce-2 space generated by $e(w)$. For j equal to $1, \ldots, n$, there exists a pure state w_j of A such that w_j lies in $x_j + U$ and J_{w_j} coincides with J_w. By [50], Theorem 4.1, $\partial_e S_*(c(A_2^{**}(e(w))))$, the set of normal pure states of $c(A_2^{**}(e(w)))$, is w*-dense in $\partial_e S(A) \cap (J_w)^\circ$. Thus, for j equal to $1, \ldots, n$, the existence of a pure state w_j of A in $(x_j + U) \cap (J_w)^\circ$ implies the existence of a pure state y_j of A in $(x_j + U) \cap \partial_e S_*(c(A_2^{**}(e(w))))$. By Proposition 7.10, the state

$$y = \sum_{j=1}^n \lambda_j y_j$$

is an element of $\partial_f^f S(A)$, lying in $U+x$ since U is convex. Since A^* is a locally convex topological vector space in the w*-topology, this shows that

$$\mathrm{conv}(\partial_e S(A) \cap K^\circ) \subseteq \overline{\partial_f^f S(A)}^{w*}.$$

Applying the Krein-Milman theorem to the w*-compact convex set $A^*_{+,1} \cap K^\circ$, (ii) follows.

(ii)⇒(iii): This is immediate since $\partial_f^f S(A)$ is contained in $\partial_f S(A)$.

(iii)⇒(i): Suppose that $J_1, \ldots, J_n$ are norm-closed ideals of A such that no J_j is contained in K. Then, for each j equal to $1, \ldots, n$, there exists an element a_j in J_j^+ such that a_j does not lie in K and x_j in $S(A) \cap K^\circ$ such that $x_j(a_j)$ is strictly positive. Define an element x in $S(A) \cap K^\circ$ by

$$x = \frac{1}{n} \sum_{j=1}^{n} x_j.$$

Then, for j equal to $1, \ldots, n$, $x(a_j)$ is strictly positive and, by hypotheses, there exists an element y in $\partial_f S(A)$ such that, for j equal to $1, \ldots, n$, J_j is not contained in $\ker y$. Since y is factorial, this implies that $(J_j^{\circ\circ})^\perp$ is contained in $k(A_0^{**}(e(y)))$. Thus

$$\begin{aligned} c(A_2^{**}(e(y))) \cap \overline{\bigcap_{j=1}^{n} J_j}^{w*} &= \bigcap_{j=1}^{n} \left[c(A_2^{**}(e(y))) \cap \overline{J_j}^{w*} \right] \\ &= \bigcap_{j=1}^{n} c(A_2^{**}(e(y))) \\ &= c(A_2^{**}(e(y))). \end{aligned}$$

Hence, $\bigcap_{j=1}^{n} J_j$ cannot be zero, and K is primal, as required. ■

In the special case of C*-algebras, Theorem 7.11 reduces to [6], Theorem 3.3.

The argument of Theorem 7.11 does not immediately translate to the JB*-triple case. The proof that (i) implies (ii) fails because the quasi-equivalent pure states with convex sum y may not lie in the same face, and thus y may not be of unit norm.

The proof that (iii) implies (i) fails because, in the absence of positivity, the sum x may be zero on some a_j. The method of solution is to reduce the problem from the JB*-triple A to an inner ideal of A which is a JB*-algebra. First, it is shown that primality is preserved when passing to inner ideals.

Proposition 7.12 *Let A be a JB*-triple and let J be a primal norm-closed ideal. Let I be a norm-closed inner ideal of A not contained in J. Then $I \cap J$ is a primal norm-closed ideal of I.*

Proof. Let $J_1, \ldots, J_n$ be norm-closed ideals of I with zero intersection. Then, by Proposition 4.6 (ii),

$$\begin{aligned}
(\bigcap_{j=1}^{n} c_n(J_j)) \cap I &= \bigcap_{j=1}^{n} (c_n(J_j) \cap I) \\
&= \bigcap_{j=1}^{n} J_j \\
&= \{0\} .
\end{aligned}$$

Thus, by Proposition 4.6, (i)

$$\begin{aligned}
\cap_{j=1}^{n} c_n(J_j) \cap c_n(I) &= c_n((\cap_{j=1}^{n} c_n(J_j)) \cap I) \\
&= \{0\} .
\end{aligned}$$

Since I is not contained in J, neither is $c_n(I)$. Since J is primal, $c_n(J_j)$ must be a subset of J for some j. Using Proposition 4.6 (ii),

$$J_j = c_n(J_j) \cap I \subseteq J \cap I.$$

Hence, $J \cap I$ is primal in I. ∎

Theorem 7.13 *Let A be a JB*-triple and let J be a primal ideal of A, with topological annihilator J°. Then, the set $\partial_f^f A_1^*$ of finite factorial functionals of A, is w^*-dense in ∂J_1°, the surface of the unit ball of J°.*

Proof. Let x be an element of ∂J_1° for which there exists an element a in A_1 be such that $x(a)$ equals 1. Let $r(a)$ be the support tripotent of a in A^{**}, let $P_2(r(a))$ be the Peirce-2 projection of $r(a)$ and let $A_2^{**}(r(a))$ be the Peirce-2 space of $r(a)$ in A^{**}. Let $I_n(a)$ be the smallest norm-closed inner ideal of A containing a. Then, by Proposition 2.22, $I_n(a)$ is a JB*-subalgebra of its bidual $A_2^{**}(r(a))$. Since $x(r(a))$ equals 1, x restricts to a state of $I_n(a)$, vanishing on $J \cap I_n(a)$. By Proposition 7.12, $J \cap I_n(a)$ is a primal ideal of the JB*-algebra $I_n(a)$. Applying Theorem 7.11, there exists a net $(x_\lambda)_{\lambda\in\Lambda}$ in $\partial_f^f S(I_n(a))$, w*-convergent to x. Clearly $(x_\lambda \circ P_2(r(a)))_{\lambda\in\Lambda}$ is a net in $\partial_f^f A_1^*$ w*-convergent to x. The result now follows by the Bishop-Phelps Theorem [14]. ■

7.4 Relationship between factorial functionals and primal ideals

Before we can consider the relationship between a functional x on a JB*-triple A and $k_n(\ker_* x)$, the norm central kernel of x in A, it is necessary to investigate how x relates to the central kernel $k(\ker^* x)$ in A^{**}. That is the purpose of this section.

Proposition 7.14 *Let x be a factorial functional on a JB*-triple A. Then the norm central kernel $k_n(\ker_* x)$ of x, is prime.*

Proof. Let $e(x)$ be the support tripotent of x, and let $c(A_2^{**}(e(x)))$ be the central hull of the Peirce-2 space of $e(x)$ in A^{**}. Since $c(A_2^{**}(e(x)))$ is a factor, $k(A_0^{**}(e(x)))$, the central kernel of the Peirce-0 space of $e(x)$, is a maximal element of the complete Boolean algebra of w*-closed ideals of A^{**}, and by Corollary 4.19 and Corollary 6.16,

$$k_n(\ker_* x) = k(A_0^{**}(e(x))) \cap A$$

is prime in A. ■

Corollary 7.15 *Let A be a JB*-triple and let $\overline{\partial_f A_1^*}^{w*}$ be the w*-closure of the set of factorial functionals of A. Let x be an element of ∂A_1^*, the surface of the dual unit ball A_1^* in the dual A^* of A, and let $k_n(\ker_* x)$ be the norm central kernel of x. Then, the following are equivalent:*

(i) *x lies in $\overline{\partial_f A_1^*}^{w*}$;*

(ii) *$k_n(\ker_* x)$ is primal.*

Proof. (i)⇒(ii): Let $J_1, \ldots, J_n$ be norm-closed ideals of A such that no J_j is contained in $k_n(\ker_* x)$. Then, for j equal to $1, \ldots, n$, there exists an element a_j in J_j such that $x(a_j)$ is non-zero. By hypothesis there exists an element y in $\partial_f A_1^*$ non-zero on each a_j. Thus no J_j is contained in $k_n(\ker_* y)$. Since, by Proposition 7.14, $k_n(\ker_* y)$ is prime, $\cap_{j=1}^n J_j$ is not contained in $k_n(\ker_* y)$. In particular, $\cap_{j=1}^n J_j$ is non-zero, and, therefore, $k_n(\ker_* x)$ is primal.

(ii)⇒(i): Let J be the primal ideal $k_n(\ker_* x)$. Then x lies in ∂J_1° and the result follows from Theorem 7.13. ■

Corollary 7.15 had previously been proved in the special case when A is a C*-algebra and x a state ([6], Section 3).

Corollary 7.16 *Let A be a JB*-triple. Then $\partial_f^f A_1^*$ is w*-dense in $\partial_f A_1^*$.*

Proof. Let x be an element of $\partial_f^f A_1^*$ and let J be the ideal $k_n(\ker_* x)$. Then x lies in ∂J_1° and by Proposition 7.14, J is prime, and, hence, primal. Applying Theorem 7.13 gives the result. ■

7.5 Additivity

Let A be a JB*-triple, let $\overline{\partial_f A_1^*}^{w*}$ be the w*-closure of the set of factorial functionals on A, let Primal A be the set of primal ideals of A and for x in $\overline{\partial_f A_1^*}^{w*}$ let $k_n(\ker x)$ be the norm central kernel of x. In the previous section, Theorem 7.13 was used to show that the map $\Psi : x \mapsto k_n(\ker x)$ maps $\overline{\partial_f A_1^*}^{w*}$ into Primal A. Before proceeding to give more applications of Theorem 7.13 in subsequent sections, some further observations about the mapping Ψ are made. It is shown that Ψ has the curious property of mapping finite sums into intersections. The equivalent results for JB*-algebras can easily be deduced using positivity, but the JB*-triple proof requires results from the theory of central hulls and central kernels developed in [41], [39] and [40].

Lemma 7.17 *Let A be a JBW*-triple with pre-dual A_* and let $\partial_e A_{*,1}$ be the set of normal pure functionals on A. For an element x in A_*, let $e(x)$ be the support tripotent, let $A_2(e(x))$ be the Peirce-2 space corresponding to $e(x)$ and let $c(A_2(e(x)))$ be the central hull of $A_2(e(x))$. Let F be a finite collection of quasi-equivalent elements of $\partial_e A_{*,1}$ and, for each x in F, let α_x be a non-zero element of $\mathbb{C}$ such that $\sum_{x\in F}\alpha_x x$ is non-zero. Then*

$$A_2(e(\textstyle\sum_{x\in F}\alpha_x x)) \subseteq \bigvee_{x\in F} A_2(e(x))$$

and

$$c(A_2(e(\textstyle\sum_{x\in F}\alpha_x x))) = c(\vee_{x\in F} A_2(e(x))).$$

is a factor.

Proof. Let K denote the w*-closed inner ideal $\vee_{x\in F} A_2(e(x))$. Then x lies in K_* for all x in F. Thus $\sum_{x\in F}\alpha_x x$ lies in K_*, $e(\sum_{x\in F}\alpha_x x)$ lies in K and

$$A_2(e(\textstyle\sum_{x\in F}\alpha_x x)) \subseteq \vee_{x\in F} A_2(e(x)).$$

Let J denote the factor $c(A_2(e(x)))$ for some, and, by quasi-equivalence, all, elements x in F. Then $e(x)$ lies in J for all x in F and

$$A_2(e(\textstyle\sum_{x\in F}\alpha_x x)) \subseteq \vee_{x\in F} A_2(e(x)) \subseteq J.$$

Since J is a factor and $\sum_{x\in F}\alpha_x x$ is non-zero,

$$c(A_2(e(\textstyle\sum_{x\in F}\alpha_x x))) = c(\vee_{x\in F} A_2(e(x))) = J,$$

thereby completing the proof. ■

Theorem 7.18 *Let A be a JBW*-triple and adopt the notation of Lemma 7.17. Let F be a finite collection of elements of $\partial_e A_{*,1}$ and for each x in F, let α_x be a non-zero element of $\mathbb{C}$ such that, for every non-empty subset E of F, $\sum_{x\in E}\alpha_x x$ is non-zero. Then,*

$$c(A_2(e(\textstyle\sum_{x\in F}\alpha_x x))) = c(\vee_{x\in F}A_2(e(x))).$$

Proof. Let Λ denote the collection of quasi-equivalence classes of F. For λ in Λ define x_λ to be $\sum_{x\in\lambda}\alpha_x x$. Define y by

$$y = \textstyle\sum_{x\in F}\alpha_x x = \sum_{\lambda\in\Lambda}x_\lambda.$$

Since the set $\{e(x_\lambda) : \lambda \in \Lambda\}$ is pairwise orthogonal, by [39], Theorem 5.4 and Lemma 5.5,

$$e(y) = \textstyle\sum_{\lambda\in\Lambda}e(x_\lambda).$$

Thus for each μ in Λ, $e(y) - e(x_\mu)$ is a tripotent orthogonal to $e(x_\mu)$. Therefore $e(x_\mu) \leq e(y)$ and, using Lemma 7.17,

$$\bigvee_{\lambda\in\Lambda} A_2(e(x_\lambda)) = A_2(e(y)) \subseteq \bigvee_{x\in F} A_2(e(x)).$$

Thus

$$c(\vee_{\lambda\in\Lambda}A_2(e(x_\lambda))) = c(A_2(e(y))) \subseteq c(\vee_{x\in F}A_2(e(x))).$$

By [40] and Lemma 7.17,

$$\begin{aligned} c(\vee_{\lambda\in\Lambda}A_2(e(x_\lambda))) &= \vee_{\lambda\in\Lambda}c(A_2(e(x_\lambda))) \\ &= \vee_{\lambda\in\Lambda}c(\vee_{x\in\lambda}A_2(e(x))) \\ &= c(\vee_{\lambda\in\Lambda}(\vee_{x\in\lambda}A_2(e(x)))) \\ &= c(\vee_{x\in F}A_2(e(x))). \end{aligned}$$

Hence,

$$c(A_2(e(y))) = c(\vee_{x\in F} A_2(e(x))),$$

as required. ■

Corollary 7.19 *Let A be a JBW*-triple with pre-dual A_* and let $\partial_e A_{*,1}$ be the set of normal pure functionals on A. For x in A_*, let $\ker x$ be the kernel of x in A and let $k(\ker x)$ be the central kernel of x in A. Let F be a finite collection of elements of $\partial_e A_{*,1}$ and for each x in F, let α_x be a non-zero element of $\mathbb{C}$ such that, for every non-empty subset E of F, $\sum_{x\in E} \alpha_x x$ is non-zero. Then,*

$$k(\ker \textstyle\sum_{x\in F} \alpha_x x) = \displaystyle\bigcap_{x\in F} k(\ker x).$$

Proof. For x in A_* let $e(x)$ be the support tripotent, let $A_2(e(x))$ be the Peirce-2 space corresponding to $e(x)$ and let $A_0(e(x))$ be the Peirce-0 space corresponding to $e(x)$. For a w*-closed subspace L of A, let $c(L)$ denote the central hull of L and let $k(L)$ denote the central kernel. Let y denote $\sum_{x\in F} \alpha_x x$. Using Corollary 4.17, Theorem 4.2, Lemma 2.8 and Theorem 7.18,

$$k(\ker y) = k(A_0(e(y))) = c(A_2(e(y)))^{\perp} = c(\vee_{x\in F} A_2(e(x)))^{\perp}.$$

Using Corollary 4.17, Theorem 4.2 and Lemma 2.8 again with [40], equation 3.2,

$$c(\vee_{x\in F} A_2(e(x)))^{\perp} = k(\cap_{x\in F} A_0(e(x))) = \bigcap_{x\in F} k(A_0(e(x))) = \bigcap_{x\in F} k(\ker x),$$

as required. ■

7.6 Prime and antiliminal JB*-triples

A JB*-triple A is said to be *prime* if the zero ideal is a prime ideal of A. The *socle* of a JB*-triple A is the norm-closed ideal $K(A)$ generated by the minimal tripotents of

A. A key step in the proof of the Glimm Stone-Weierstrass theorem for JB*-triples ([67], Theorem 6.2) is the following result.

Theorem 7.20 ([21], Theorem 5.6) *Let A be a non-zero JB*-triple, let A_1^* be the dual unit ball of A and let $\partial_e A_1^*$ be the set of pure functionals of A. Then $\partial_e A_1^*$ is w^*-dense in A_1^* if and only if A is an infinite dimensional Hilbert space, an infinite dimensional spin factor or A is prime with zero socle.*

Let A be a JB*-triple with bidual A^{**}. For each pure functional x, let $c(A_2^{**}(e(x)))$ be the central hull of the Peirce-2 space of the support tripotent $e(x)$ of x in A^{**} and let $(\pi_x, c(A_2^{**}(e(x))))$ be the Cartan factor representation corresponding to x. The JB*-triple A is said to be *liminal* if $K(c(A_2^{**}(e(x))))$, the socle of $c(A_2^{**}(e(x)))$, coincides with $\pi_x(A)$ for each pure functional x of A, and *postliminal* if $K(c(A_2^{**}(e(x))))$ is a subset of $\pi_x(A)$ for each pure functional x of A. The JB*-triple A is said to be *antiliminal* if it possesses no non-zero liminal ideals. The JB*-triple A is antiliminal if and only if it possesses no abelian elements.

In a prime JB*-triple, an element is abelian if and only if it is a scalar multiple of a minimal tripotent ([21], Remark 3.6). Thus the condition that A is prime with zero socle is equivalent to the condition that A is prime and antiliminal. In the light of Theorem 7.20, it is natural to ask whether factorial functionals can be used to separately characterise the prime JB*-triples and the antiliminal JB*-triples. In the C*-algebra setting, this question has already been answered ([6], Corollary 3.4, [12], Theorem 3.4). For JB*-triples, the preceding results on factorial functionals may be applied to obtain the following characterisation.

Proposition 7.21 *Let A be a JB*-triple, let $\partial_f A_1^*$ be the set of factorial functionals and let $\partial_f^f A_1^*$ be the set of factorial functionals of Type* $\mathrm{I_n}$ *where n is finite. Then the following conditions are equivalent:*

(i) *A is prime;*

(ii) $\partial A_1^* \subseteq \overline{\partial_f^f A_1^*}^{w*}$;

(iii) $\partial A_1^* \subseteq \overline{\partial_f A_1^*}^{w*}$.

Proof. (i)⇒(ii) Take J to be the zero ideal in Theorem 7.13.

(ii)⇒(iii) This is immediate.

(iii)⇒(i) Let A be such that $\partial A_1^* \subseteq \overline{\partial_f^f A_1^*}^{w*}$ and assume that J_1 and J_2 are non-zero norm-closed ideals with zero intersection. Then, there exists an element x in ∂A_1^* non-zero at elements a_1 in J_1 and a_2 in J_2. By hypothesis, there exists an element y in $\partial_f^f A_1^*$ non-zero at a_1 and a_2. Thus J_1 and J_2 are not subsets of the prime ideal $k_n(\ker y)$ (Proposition 7.14) thereby giving a contradiction. ■

In the remainder of this section the relationship between antiliminality and the set of factorial functionals is investigated. Lemma 7.22 is stated in the discussion following [19], Lemma 3.4.

Lemma 7.22 *Let A be a JB*-triple and let $\partial_e A_1^*$ be the set of pure functionals of A. For each element x in $\partial_e A_1^*$, let $e(x)$ be the support tripotent of x, let $K(c(A_2^{**}(e(x))))$ be the socle of the central kernel of the Peirce-2 space of $e(x)$ in A^{**}, let $k(A_0^{**}(e(x)))$ be the central kernel of the Peirce-0 space of $e(x)$ in A^{**}, let π_x be the canonical homomorphism of A into $c(A_2^{**}(e(x)))$ and let C_x be the norm-closed ideal of A defined by*

$$C_x = (K(c(A_2^{**}(e(x)))) \oplus k(A_0^{**}(e(x)))) \cap A.$$

Then

$$\bigcap_{x \in \partial_e A_1^*} C_x = \{a \in A : \pi_x(a) \in K(c(A_2^{**}(e(x))))\, \forall x \in \partial_e A_1^*\}$$

and this set is the largest liminal ideal of A.

Lemma 7.23 *Let A be an antiliminal JB*-triple and adopt the notation of Lemma 7.22. Then*

$$\overline{\bigcup_{x \in \partial_e A_1^*} C_x^\circ \cap \partial A_1^*}^{w*} = \overline{\partial_e A_1^*}^{w*}.$$

Proof. Since A is antiliminal, $\bigcap_{x \in \partial_e A_1^*} C_x$ is zero. The result now follows as in [21], Corollary 3.8. ■

Lemma 7.24 *Let A be an antiliminal JB*-triple, let $\partial_e A_1^*$ be the set of pure functionals of A, let x be an element of $\partial_e A_1^*$, let J_x be the norm central kernel of x in A and let $(J_x)^\circ$ be the topological annihilator of J_x. Then $(J_x)^\circ \cap \partial A_1^*$ is a subset of $\overline{\partial_e A_1^*}^{w*}$.*

Proof. The argument is exactly as in the proof of [21], Theorem 5.6. ∎

Proposition 7.25 *Let A be an antiliminal JB*-triple, let $\partial_e A_1^*$ be the set of pure functionals of A and let $\partial_f A_1^*$ be the set of factorial functionals of A. Then $\partial_e A_1^*$ is w*-dense in $\partial_f A_1^*$.*

Proof. Let x be an element of $\partial_f^f A_1^*$. Then by Proposition 7.10, x is a convex combination of quasi-equivalent elements $x_1, \ldots, x_n$ of $\partial_e A_1^*$. Let J_{x_1} be the norm central kernel of x_1. For j equal to $2, \ldots, n$, x_j is quasi-equivalent to x_1, so J_{x_1} is also the norm central kernel of x_j. It follows that x is an element of $J_{x_1}^\circ \cap \partial A_1^*$ and, by Lemma 7.24, of $\overline{\partial_e A_1^*}^{w*}$. The result follows from Corollary 7.16. ∎

A similar result relating pure states to factorial states is known for C*-algebras ([12], Proposition 3.1). Antiliminality is not a necessary condition for $\partial_e A_1^*$ to be w*-dense in $\partial_f A_1^*$. By Example 7.4, [21], Corollary 4.3, and the argument of [12], Theorem 3.4, if A is a JB*-triple possessing an abelian ideal J such that A/J is antiliminal, an infinite dimensional Hilbert space or an infinite dimensional spin triple, then $\partial_e A_1^*$ is w*-dense in $\partial_f A_1^*$. It is not known if these sufficient conditions are necessary.

7.7 Restriction of functionals to the centralizer

Let A be a JB*-triple with centralizer $Z(A)$. In this section a map $x \mapsto \check{x}$ is defined from A^*, the dual of A, to $Z(A)^*$, the dual of $Z(A)$. The map extends the map from $\partial_e A_1^*$, the set of pure functionals of A, to $\Omega(Z(A))$, the character space of $Z(A)$, given in Theorem 3.22. The map defined in this section is used to characterise the pure functionals of a continuous JBW*-triple in terms of the pure states of its centralizer in Section 7.8.

Lemma 7.26 *Let A be a JB*-triple with dual A^* and bidual A^{**}. Let $Z(A)$ be the centralizer of A and let $Z(A)^*_+$ be the positive cone in the dual of $Z(A)$. Let x be an element of A^* with support tripotent $e(x)$ in A^{**}. Define the functional $\check{x}$ on $Z(A)$, for T in $Z(A)$, by*

$$\check{x}(T) = x(T^{**}e(x)).$$

Then $\check{x}$ is an element of $Z(A)^*_+$ such that

$$\|\check{x}\| = \|x\|.$$

Proof. Clearly $\check{x}$ is a linear functional on $Z(A)$ bounded by $\|x\|$. Let Id_A be the unit of $Z(A)$. Since

$$\check{x}(\mathrm{Id}_A) = x(e(x)) = \|x\|,$$

$\check{x}$ is positive on $Z(A)$ with norm $\|x\|$. ■

Lemma 7.27 shows how Lemma 7.26 is connected to the process of restricting positive functionals to the centre in the unital C*-algebra case.

Lemma 7.27 *Let A be a unital C*-algebra with unit 1, dual A^* and centralizer $Z(A)$. Let x be an element of A^*, let $|x|$ be the absolute value of x and let $\check{x}$ be the functional defined on $Z(A)$ as in Lemma 7.26. Then, for T in $Z(A)$,*

$$\check{x}(T) = |x|(T1).$$

Proof. By w*-continuity of multiplication in A^{**}, the bidual of A, for a in A^{**},

$$T^{**}a = (T1)a.$$

Therefore, by Corollary 2.18,

$$\check{x}(T) = x(T^{**}e(x)) = x(e(x)T1) = |x|(T1),$$

as required. ■

Lemma 7.28 *Let A be a JB*-triple with dual A^* and bidual A^{**}, let x be an element of A^* with support tripotent $e(x)$ in A^{**} and let $\check{x}$ be the functional defined on $Z(A)$ as in Lemma 7.26. Let ϕ_x be the sesquilinear map defined for a and b in A by*

$$\phi_x(a, b) = x(\{a\, b\, e(x)\}).$$

Then, for each element T in the centralizer, $Z(A)$, of A, with adjoint $T^\dagger$, and each element a in A,

$$|x(Ta)|^2 \leq \check{x}(TT^\dagger)\phi_x(a, a).$$

When x lies in the set $\partial_f A_1^*$ of factorial functionals of A,

$$x(Ta) = \check{x}(T)x(a).$$

Proof. Let x be an element of A^*, let T be an element of $Z(A)$ and let a be an element of A. Writing a_2 for $P_2(e(x))a$, and $a_2^{*e(x)}$ for $\{e(x)\, a_2\, e(x)\}$,

$$x(Ta) = x(P_2(e(x))T^{**}a) = x(T^{**}\{e(x)\, a_2^{*e(x)}\, e(x)\}) = \phi_x(T^{**}e(x), a_2^{*e(x)}).$$

Now,

$$\begin{aligned} \phi_x(T^{**}e(x), T^{**}e(x)) &= x(\{T^{**}e(x)\, T^{**}e(x)\, e(x)\}) \\ &= x(T^{**}(T^{**})^\dagger e(x)) \\ &= \check{x}(TT^\dagger), \end{aligned}$$

and,

$$\begin{aligned} \phi_x(a_2^{*e(x)}, a_2^{*e(x)}) &= x(\{\{e(x)\, a_2\, e(x)\}\, \{e(x)\, a_2\, e(x)\}\, e(x)\}) \\ &= x(\{e(x)\, a_2\, P_2(e(x))a_2\}) \\ &= \phi_x(a_2, a_2) \\ &\leq \phi_x(a, a). \end{aligned}$$

Therefore, by the Schwarz inequality,

$$|x(Ta)|^2 \leq \check{x}(TT^\dagger)\phi_x(a,a).$$

Now let x be an element of $\partial_f A_1^*$. Then $A_2^{**}(e(x))$ is a factor, and, hence, for T in $Z(A)$,

$$T^{**}e(x) = \check{x}(T)e(x).$$

Therefore, using the commutativity of T^{**} with $P_2(e(x))$,

$$\begin{aligned} x(Ta) &= x(T^{**}P_2(e(x))a) \\ &= x(\{T^{**}e(x)\,\{e(x)\,a\,e(x)\}\,e(x)\}) \\ &= \check{x}(T)x(a), \end{aligned}$$

as required. ■

In particular, Lemma 7.28 shows that when A is a JB*-triple and x a pure functional, the definition of $\check{x}$ given in Lemma 7.26 agrees with the definition of $\check{x}$, given in Proposition 3.22.

Lemma 7.29 *Let A be a JB*-triple, let x be an element of A^* with support tripotent $e(x)$ in A^{**} and let $\check{x}$ be the functional on the centralizer $Z(A)$ of A defined in Lemma 7.26. Let $k_n(\ker x)$ denote the norm central kernel of x in A and let $\ker \check{x}$ denote the kernel of $\check{x}$ in $Z(A)$. Then, the following results hold.*

(i) *Let I be an ideal of $Z(A)$ such that $\check{x}$ annihilates I and let $I_\triangle$ be the norm-closed ideal*

$$I_\triangle = IA$$

of A. Then

$$I_\triangle \subseteq k_n(\ker x).$$

(ii) *Let J be an ideal of A such that x annihilates J and let $J^\triangledown$ be the ideal*

$$J^\triangledown = \{T \in Z(A) : TA \subseteq J\}$$

of $Z(A)$. Then

$$J^\triangledown \subseteq \ker \check{x}.$$

Proof. (i) For T in I with adjoint $T^\dagger$, $TT^\dagger$ lies in I, and, by Lemma 7.28, for all a in A, $x(Ta)$ is zero. The result follows.

(ii) For T in $J^\triangledown$ and a in A, x is zero at Ta. Let (a_λ) be a net in A, w*-convergent to $e(x)$ in A^{**}. Then (Ta_λ) is a net in A w*-convergent to $T^{**}e(x)$ and it follows that $x(T^{**}e(x))$ is zero. Therefore, T lies in $\ker \check{x}$. ■

7.8 Pure functional space of a continuous JBW*-triple

A JBW*-triple is said to be *continuous* if it contains no w*-closed ideal which is a Type I JBW*-triple. In this section the preceding results are applied to obtain a characterisation of the pure functional space of a continuous JBW*-triple in terms of the pure state space of the centralizer.

Proposition 7.30 *Let A be a JBW*-triple with dual A^* and centralizer $Z(A)$, and let $\partial_f A_1^*$ be the set of factorial functionals of A. Let x be an element of ∂A_1^*, the surface of the unit ball of A^*, let $k_n(\ker x)$ be the norm central kernel of x and let $\check{x}$ be the functional defined on $Z(A)$ as in Lemma 7.26. Then, the following are equivalent:*

(i) *x lies in $\overline{\partial_f A_1^*}^{w*}$;*

(ii) *$k_n(\ker x)$ is primal;*

(iii) *$k_n(\ker x)$ contains a Glimm ideal;*

(iv) *$\check{x}$ is a character of $Z(A)$.*

Proof. (i)⇔(ii): This follows from Corollary 7.15.

(ii)⇔(iii): This is a particular case of Proposition 6.14.

(iii)⇒(iv): Let ω be a character of $Z(A)$ with kernel I_ω in $Z(A)$, such that the Glimm ideal

$$K_\omega = (I_\omega)_\triangle = I_\omega A$$

is annihilated by x. Let $K_\omega^\triangledown$ be the norm-closed ideal

$$K_\omega^\triangledown = \{T \in Z(A) : TA \subseteq K_\omega\}$$

of $Z(A)$. Then, by Lemma 7.29,

$$I_\omega \subseteq K_\omega^\triangledown \subseteq \ker \check{x}.$$

For T in $Z(A)$, $T - \omega(T)I$ lies in I_ω and hence,

$$\check{x}(T) = \check{x}(T - \omega(T)I) + \omega(T)\check{x}(I) = \omega(T).$$

Thus $\check{x}$ coincides with ω and $\check{x}$ is a character of $Z(A)$.

(iv)⇒(iii): Suppose that $\check{x}$ is a character of $Z(A)$ and let $I_{\check{x}}$ be the kernel of $\check{x}$ in $Z(A)$. Then, by Lemma 7.29, the Glimm ideal $I_{\check{x}}A$ is contained in $k_n(\ker x)$. ■

The characterisation of the pure functional space of a continuous JBW*-triple may now be given.

Theorem 7.31 *Let A be a continuous JBW*-triple with dual space A^* and centralizer $Z(A)$. Let ∂A_1^* be the surface of the dual unit ball, let $\partial_e A_1^*$ be the set of extreme points of the dual unit ball and let $\partial_e S((Z(A)))$ be the set of extreme points of the state space of $Z(A)$. For x in A^*, let $\check{x}$ be the functional defined on $Z(A)$ in Lemma 7.26. Then*

$$\overline{\partial_e A_1^*}^{w*} \cap \partial A_1^* = \{x \in \partial A_1^* : \check{x} \in \partial_e S(Z(A))\}.$$

Proof. Let a be an abelian element of A. Then, the w*-closed inner ideal generated by a is a commutative W*-algebra, generated by abelian tripotents of A. Hence, any abelian element of A is zero and A is antiliminal. By Proposition 7.25, if A is an antiliminal JB*-triple then

$$\overline{\partial_e A_1^*}^{w*} \cap \partial A_1^* = \overline{\partial_f A_1^*}^{w*} \cap \partial A_1^*.$$

The result is now immediate from Proposition 7.30. ■

In the W*-algebra case, Theorem 7.31 is an order-free analogue of a characterisation of the pure state space [49], [6].

Index

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